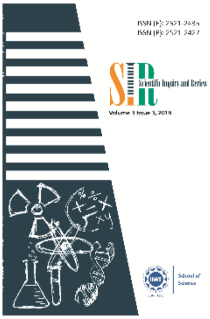

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Author(s): Abdul Majeed[1], Mohsin Kamran[1], Muhammad Abbas[2]

Affiliation: [1]Department of Mathematics, Division of Science and Technology University of Education Lahore, Pakistan
[2]Department of Mathematics, University of Sargodha, Sargodha, Pakistan

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Construction of Polynomial Spiral Segment Using Cubic Ball Basis Functions

Abdul Majeed[1*], Mohsin Kamran[1], and Muhammad Abbas[2]

[1]Department of Mathematics,
Division of Science and Technology University of Education Lahore, Pakistan
[2]Department of Mathematics, University of Sargodha, Sargodha, Pakistan

*abdulmajeed@ue.edu.pk

ABSTRACT

Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) applications use Bezier, Ball curves, B-splines, and non-uniform rational B-splines (NURBS) for fairing curves. Unfortunately, in many instances, the fairness of curves is not satisfactory. Hence, spiral curves were used to design an improved form of curves known as fair curves. These fair curves are suitable for several sophisticated applications such as designing the routes of high ways and railways as well as mobile robot trajectories. This study attempted to develop a polynomial cubic Ball spiral segment with a degree of freedom. It also observed the outcome of its shape parameters. The findings of the study are presented in a graphical form.

Keywords: cubic ball basis functions, curvature, fair curves, polynomial spiral segment

INTRODUCTION

Continuous curves with undesirable curvature extrema are attributed to specific applications such as for designing the trajectories of mobile robot and high ways or railway routes [1, 2]. Such curves are called fair curves [3]. These curves are equally important in Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) applications such as [3, 4]. B-spline, Bezier, Ball curves, and non-uniform rational B-splines (NURBS) are also used in CAD and CAGD applications. However, their fairness may not be guaranteed. Polynomial spirals were developed to resolve this problem.

We studied spiral in terms of a curved line segment having variation of a signed curvature in a monotonic way. Spiral segment may refer to any segment of a curve that exists between two consecutive curvature extrema, between the first endpoint and first curvature extremum, between the last endpoint, or the last curvature extremum. A great deal of work has been done on Bezier spiral. In one of such studies, the planar cubic Bezier curve was developed [5] and was examined [6, 7] to have point of zero curvature at one end point. It was also noted that cubic Spiral segments without any point of zero curvature also exists [8] and are used for shape control [9].

In this paper, we have proposed a polynomial spiral segment (cubic Ball spiral) using cubic Ball basis functions by incorporating two spiral segment options (zero curvature at one end or non-zero curvature at both ends). The advantage of Ball basis functions can be broadly divided in two, when comparing it with Bezier basis functions.

Initially, to evaluate the Ball curve, a robust algorithm, that is suitable for the interactive design environment, was established [10]. Furthermore, a generalized Ball basis is much suited to study degree elevation and reduction since it eases data portability and curve approximation in CAD systems [11, 12].

The first section of this study explained the notation and conventions used in this work. The second section described cubic Ball curves. The development of the cubic Ball spiral is accomplished in the third section, which is followed by a section that deals with graphical presentation and conclusions.

2. NOTATION AND CONVENTIONS

We took a typical Cartesian coordinate system with x-axis and y-axis, whose direction of angle measurement is anticlockwise, Its vectors and points are represented as $\overrightarrow{W}$. Here, the order pair notation can also be used to represent points and vectors, e.g, (x, y). The components of a vector may be designated as (u_x, u_y), or for a subscripted vector, e.g, $\overrightarrow{W}_{b,0}$ as $(W_{0,x}, W_{0,y})$ or $(W_{b,0,x}, W_{b,0,y})$ for a doubly subscripted vector, $\overrightarrow{W}_{b,0}$. The dot product of two vectors, $\vec{u}$ and $\vec{v}$ is denoted as $\vec{u}\cdot\vec{v}$. The vector norm or length $\vec{u}$ is denoted as $\parallel \vec{u} \parallel = \sqrt{\vec{u}\cdot\vec{u}}$. The derivative of a function (scalar or vector valued) is denoted with a prime, e.g $\vec{P}'(\psi)$. The set of points $\vec{P}(\psi) = (X(\psi), Y(\psi))$ for real ψ defines a planar parametric curve. The symbol × is used to denote the signed z-component of the usual three-dimensional cross-product of two vectors in the xy plane, namely i.e, $\vec{u}\times\vec{v} = u_x v_y - u_y v_x$. For a plane parametric curve, the tangent vector $\vec{P}'(\psi)$ is represented by

$\vec{P}'(\psi) = (X'(\psi), Y'(\psi))$. If $\vec{P}'(\psi) \neq \vec{0} = (0,0)$, then the signed curvature of $\vec{P}(\psi)$ is defined as [3]

$$c(\psi) = \frac{\vec{P}'(\psi) \times \vec{P}''(\psi)}{\left\|\vec{P}'(\psi)\right\|^3}. \tag{2.1}$$

Differentiation of **Eq.** (**2.1**) yields

$$c'(\psi) = \frac{w(\psi)}{\left\|\vec{P}'(\psi)\right\|^5}. \tag{2.2}$$

where,

$$\begin{aligned} w(\psi) = &\{\vec{P}'(\psi) \cdot \vec{P}'(\psi)\}\{\vec{P}'(\psi) \times \vec{P}'''(\psi)\} \\ &- 3\{\vec{P}'(\psi) \times \vec{P}''(\psi)\}\{\vec{P}'(\psi) \cdot \vec{P}''(\psi)\}. \end{aligned} \tag{2.3}$$

3. CUBIC BALL BASIS CURVE

The cubic Ball polynomial basis was first proposed by Ball [12] for CAD systems application. The Ball basis functions are defined as

$$\begin{aligned} S_0(\psi) &= (1-\psi)^2, & S_1(\psi) &= 2\psi(1-\psi)^2, \\ S_2(\psi) &= 2\psi^2(1-\psi), & S_3(\psi) &= \psi^2. \end{aligned} \tag{3.1}$$

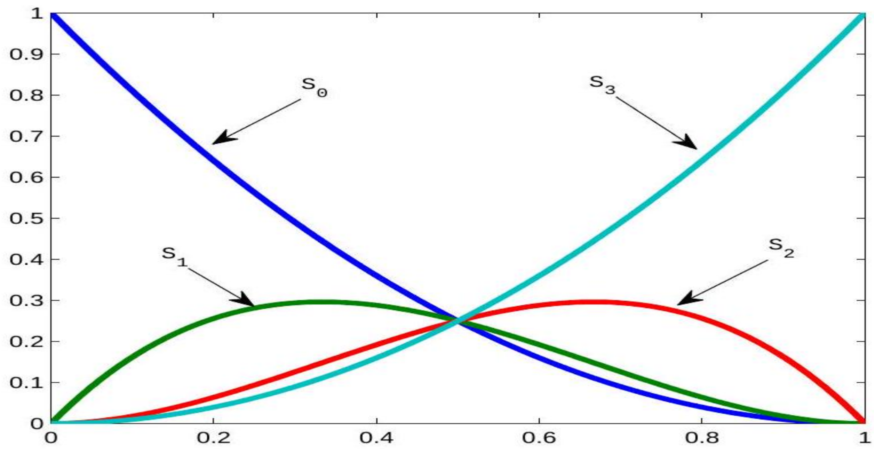


Figure 1. Ball basis functions

Figure 1 illustrates these functions against its parameter ψ. The cubic Ball curve $P(\psi)$ with control points $\overrightarrow{A_i}$ is defined as

$$P(\psi) = \vec{A}_0(1-\psi)^2 + 2\overrightarrow{A_1}(1-\psi)^2\psi + 2\overrightarrow{A_2}(1-\psi)\psi^2 + \overrightarrow{A_3}\psi^2, 0 \leq \psi \leq 1 \tag{3.2}$$

The Ball basis functions and curve obey all the properties of curve like linearly independent, non-negativity, symmetric, monotonicity, partition of unity, convex hull property and affine under linear transformation.

4. CUBIC BALL SPIRAL

In **Eq.** (**3.2**) assume the control points $\vec{A}_0, \vec{A}_1, \vec{A}_2$ and $\vec{A}_3$ are distinct. Without loss of generality, translate, rotate, and if necessary, reflect the curve such that $\vec{A}_0$ is at the origin, $\vec{A}_1$ is on the positive x-axis and $\vec{A}_3$ is above the x-axis, **Eq.** (**3.2**) may now be written as

$$\vec{P}(\psi) = (X(\psi), Y(\psi)) \tag{4.1}$$

where,

$$\begin{aligned} X(\psi) &= 2\mathrm{ag}\,\psi(1-\psi)^2 + 2(g + k\cos(\theta) - bk\cos(\theta))\psi^2(1 \\ &\quad - \psi) + (g + k\cos(\theta))\psi^2, \\ Y(\psi) &= 2(k\sin(\theta) - bk\sin(\theta))\psi^2(1-\psi) + (k\sin(\theta))\psi^2. \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\|\vec{A}_1 - \vec{A}_0\| > 0, \\ &\|\vec{A}_2 - \vec{A}_1\| > 0, \\ &\|\vec{A}_3 - \vec{A}_2\| > 0 \end{aligned} \tag{4.3}$$

where θ is a angle between $\vec{A}_3 - \vec{A}_2$ and x-axis. The curvature at $\psi = 0$ and $\psi = 1$ using **Eq.** (**2.1**) is

$$c(0) = \frac{(3-2b)k\sin(\theta)}{2a^2g^2},$$

and

$$c(1) = \frac{(3-2a)g\sin(\theta)}{2b^2k^2}.$$

Theorem 1. The curvature of $\vec{P}(\psi)$ is 0 at 0 iff $b = 3/2$.

Now by using the above theorem, **Eq.** (**4.2**) becomes

$$X(\psi) = 2ag\psi(1-\psi)^2 + 2(g - 1/2k\cos(\theta))\psi^2(1-\psi) + (g + k\cos(\theta))\psi^2 \tag{4.4}$$

$$Y(\psi) = -k\sin(\theta))\psi^2(1-\psi) + k\sin(\theta))\psi^2. \tag{4.5}$$

and curvature becomes

$$\begin{aligned} c(0) &= 0, \\ c(1) &= \frac{2(3-2a)g\sin(\theta)}{9k^2}. \end{aligned}$$

The derivatives of **Eq. (4.4)** are

$$\begin{aligned} X'(\psi) = &\ 2ag(1-\psi)^2 - 4ag(1-\psi)\psi + 4(1-\psi)\psi(g - 1/2k\cos(\theta)) \ldots \\ &-2\psi^2(g - 1/2k\cos(\theta)) + 2\psi(g + k\cos(\theta)) \\ X''(\psi) = &\ -8ag(1-\psi) + 4ag\psi + 4(1-\psi)\left(g - \frac{1}{2k\cos(\theta)}\right) - 8\psi(g - 1/2k\cos(\theta)) + 2(g + k\cos(\theta)) \\ X'''(\psi) = &\ 12ag - 12\left(g - \frac{1}{2k\cos(\theta)}\right) \end{aligned} \tag{4.6}$$

$$\begin{aligned} Y'(\psi) = &\ 2k\psi\sin(\theta) - 2k(1-\psi)\psi\sin(\theta) + k\psi^2\sin(\theta) \\ Y''(\psi) = &\ 2k\sin(\theta) - 2k(1-\psi)\sin(\theta) + 4k\psi\sin(\theta) \\ Y'''(\psi) = &\ 6k\sin(\theta) \end{aligned} \tag{4.7}$$

Using **Eq. (4.4)**, **Eq. (4.6)** and **Eq. (4.7)** we can write

$$\begin{aligned} \vec{P}'(\psi)\cdot\vec{P}'(\psi) = &\ (9k^2)\psi^4 + (+36gk\cos(\theta) - 8agk\cos(\theta))\psi^3(1-\psi) \ldots \\ &+(36g^2 - 32ag^2 + 12agk\cos(\theta))\psi^2(1-\psi)^2 + \cdots \\ &(24ag^2 - 16a^2g^2)\psi(1-\psi)^3 + 4a^2g^2(1-\psi)^4 \\ \vec{P}'(\psi)\cdot\vec{P}''(\psi) = &\ (-18gk\cos(\theta) + 18k^2 + 12agk\cos(\theta))\psi^3 + (+46gk\cos(\theta) \ldots \\ &-4k^2 + 32ag^2 - 40g^2 - 24agk\cos(\theta))\psi^2(1-\psi) + (32g^2 - 84ag^2 - 8gk \\ &\cos(\theta) \ldots - 4k^2 + 40a^2g^2 + 12agkC\cos(\theta))\psi(1-\psi)^2 + (12ag^2 - 16a^2g^2)(1-\psi)^3 \\ \vec{P}'(\psi)\times\vec{P}''(\psi) = &\ (18gk\sin(\theta) - 8agk\sin(\theta))\psi^2 - 4agk\psi^3\mathrm{Sin}(\theta) \ldots \\ &+8agk\psi(1-\psi)\mathrm{Sin}(\theta) + 4agk(1-\psi)^2\mathrm{Sin}(\theta) - 4agk(1-\psi)^3\,\mathrm{Sin}(\theta) \\ \vec{P}'(\psi)\times\vec{P}'''(\psi) = &\ (36gk\sin(\theta) - 24agk\sin(\theta))\psi - 12agk\psi^2\sin(\theta) + 12agk(1-\psi)^2\sin(\theta) \end{aligned} \tag{4.8}$$

Substitution of **Eq. (4.8)** into **Eq. (2.3)**, followed by some algebraic maipulation yields $w(\psi) = 12gkf(\psi)\sin(\theta)$ where

$$f(\psi) = \sum_{i=0}^{6} f_i(1-\psi)^{6-i}\psi^i,\ 0 \le \psi \le 1 \tag{4.9}$$

with

$$f_0 = 4a^3g^2 \tag{4.10}$$

$$f_1 = 24a^3g^2 \tag{4.11}$$

$$f_2 = -2a((21 + 2a(-94 + 25a))g^2 - 6k^2 + 12(-1 + a)gk\mathrm{Cos}(\theta)) \tag{4.12}$$

$$f_3 = 2(-2(9 + a(-87 + 14a))g^2 + 3(3 + 2a)k^2 + 2(9 + 10(-3 + a)a)gk\mathrm{Cos}(\theta)) \tag{4.13}$$

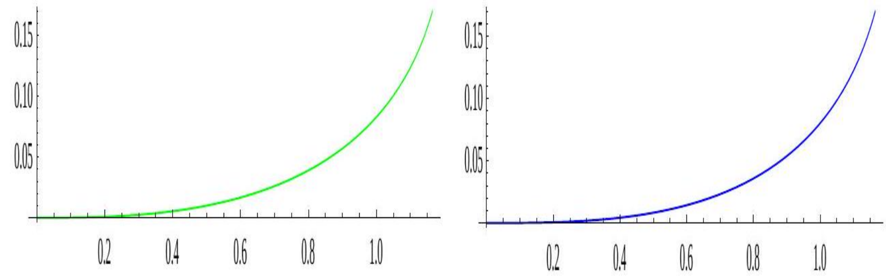
$$\begin{aligned} f_4 = {} & 6(24 + a(21 + 4a(-14 + 5a))g^2 + 3(12 - 19a)k^2 \\ & + (-63f_4 \\ & = 6(24 + a(21 + 4a(-14 + 5a))g^2 + 3(12 \\ & - 19a)k^2 + (-63 + 4(6 - 5a)a)gk\mathrm{Cos}(\theta) \\ & + 4(6 - 5a)a)gk\mathrm{Cos}(\theta) \end{aligned} \tag{4.14}$$

$$f_5 = -6(-2(-3 + 2a)(-5 + 4a)g^2 + (6 + 5a)k^2 + (3 + 2a(-5 + 4a))gk\mathrm{Cos}(\theta)) \quad (4.15$$

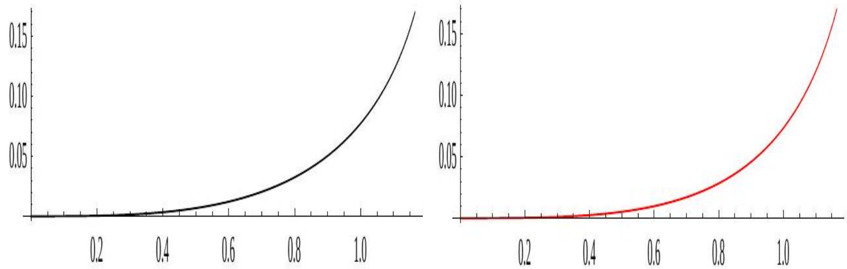
$$f_6 = 9k(3(-2 + a)k + (3 - 2a)^2 g\mathrm{Cos}(\theta)) \tag{4.16}$$

5. RESULTS AND DISCUSSION

This section illustrated the different possibilities of cubic Ball spiral curve using proposed Cubic Ball spiral. As the proposed curve has 5 degrees of freedoms, namely *a, b, g, k,* and θ. In Theorem 1 we fixed $b = 3/2$ to get the zero curvature at 0. We also constructed the different possible Ball spiral curves. In **Figure 2**, we fixed the $\theta = \pi/4$, $g = 0.996$ and $k = 0.24$ randomly and changed the free parameter a. **Figure 3** shows that curvature of Ball spiral curves was monotone. Similarly, **Figure 4** and **Figure 5** shows the different curves with their monotone curvature graphs. **Figures 6** and **7** displays the comparison between different Ball spiral curves and curvature, while keeping the values of *g, θ,* and *k* constant and changing the value of *a*. **Figure 8** presents the different Ball spiral curves with different angles. **Figure 9**(**a**) shows a cubic Ball curve, while **Figure 9**(**b**) displays the curvature of Ball curve. From **Figure 9**, we can conclude that the simple cubic Ball curve looks smooth but it is not fair since the curvature graph was not monotone.

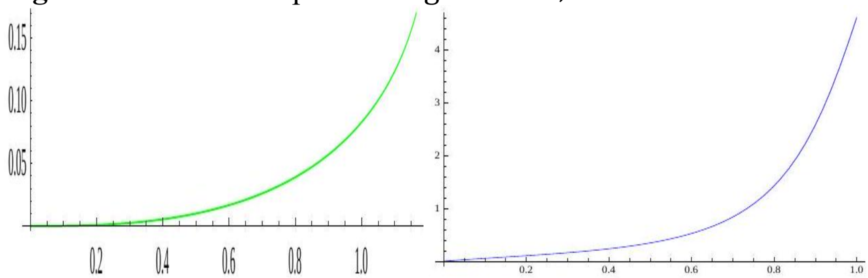


(a) a $= 0.54$ (b)$a = 0.654$

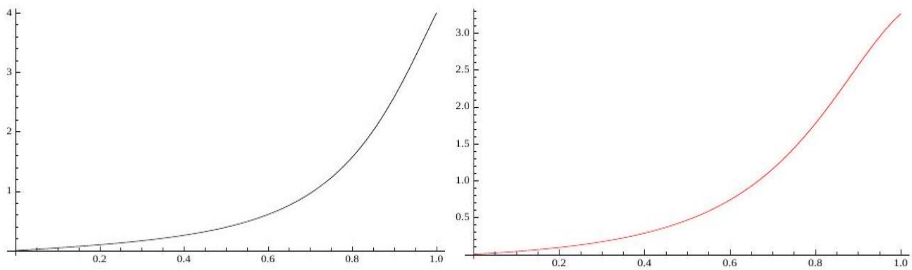


(c) $a = 0.754$ (d) $a = 0.9$

Figure 2. Cubic Ball spiral with g $= 0.996$, k $= 0.24$

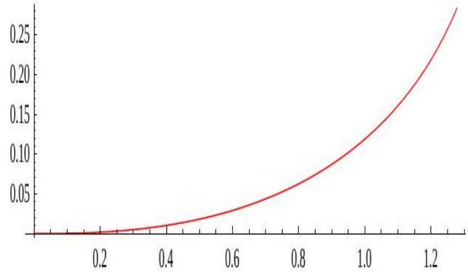


(a) $a = 0.54$ (b) $a = 0.654$

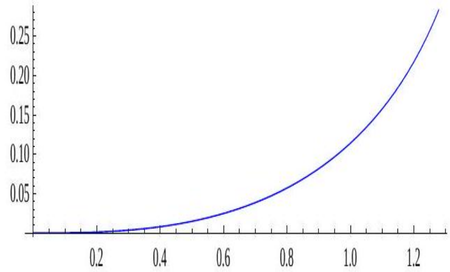


(c) a $= 0.754$ (d) a $= 0.9$

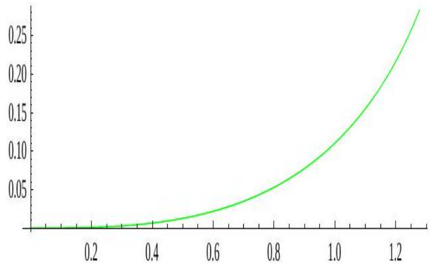
Figure 3. Monotone curvature with g $= 0.996$, k $= 0.24$



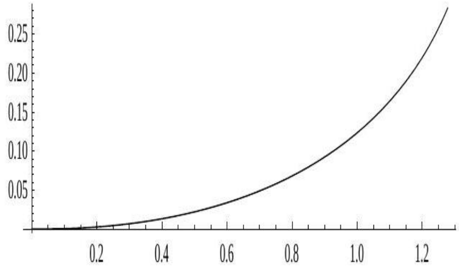
(a) $a = 0.459$



(b) $a = 0.59$

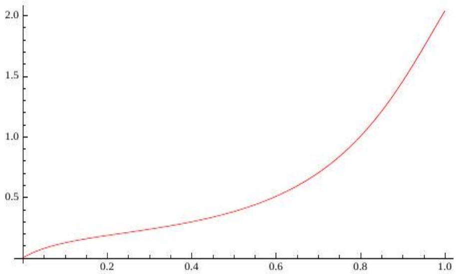


(c) $a = 0.69$

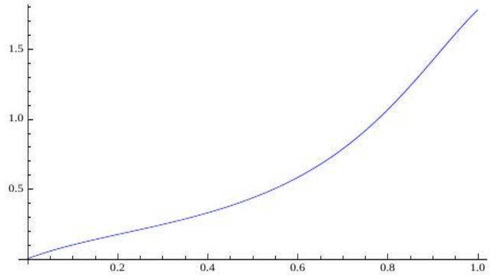


(d) $a = 0.312$

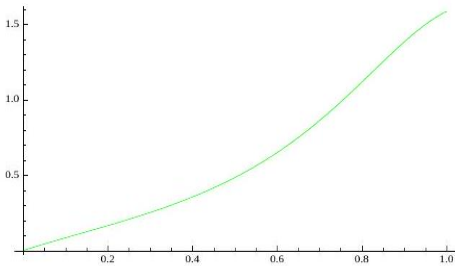
Figure 4. Cubic Ball spiral with $\mathrm{g} = 0.996, \mathrm{k} = 0.4$



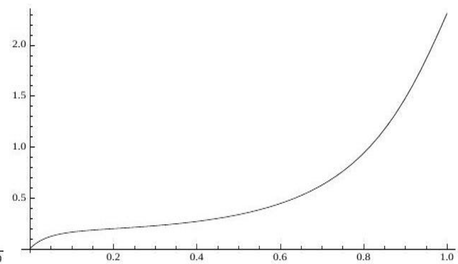
(a) $a = 0.45$



(b) $a = 0.59$



(c) $a = 0.69$



(d) $\mathrm{a} = 0.312$

Figure 5. Monotone curvature with $\mathrm{g} = 0.996, \mathrm{k} = 0.4$

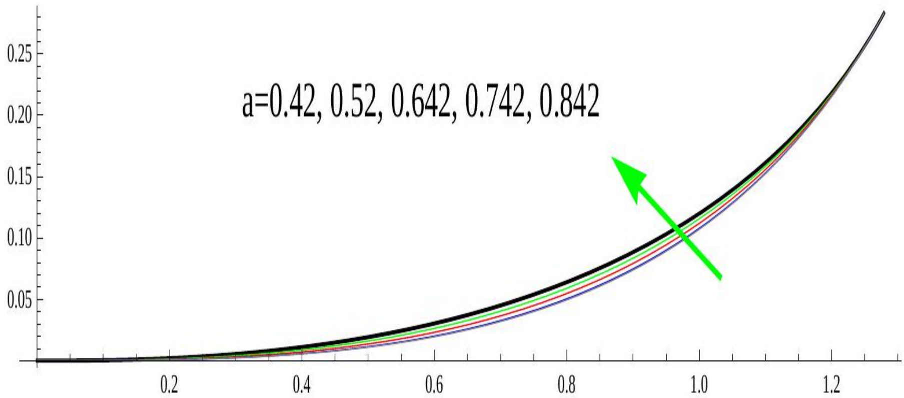


Figure 6. Ball spiral curves with different Values of a

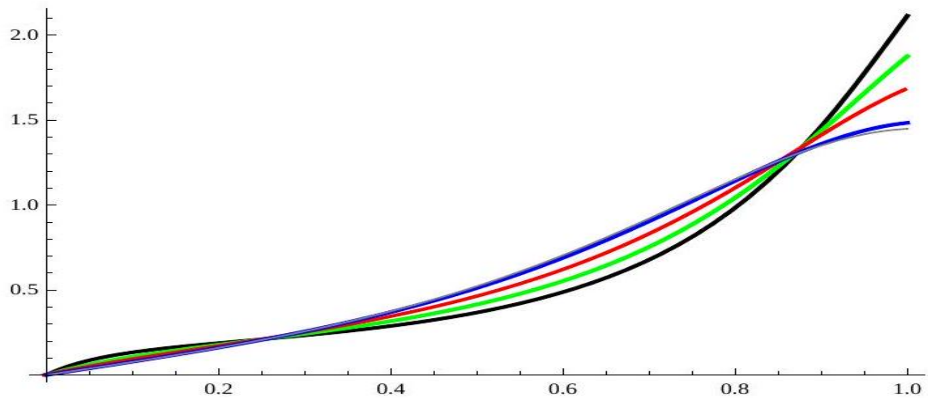


Figure 7. Curvature graph with different values of a

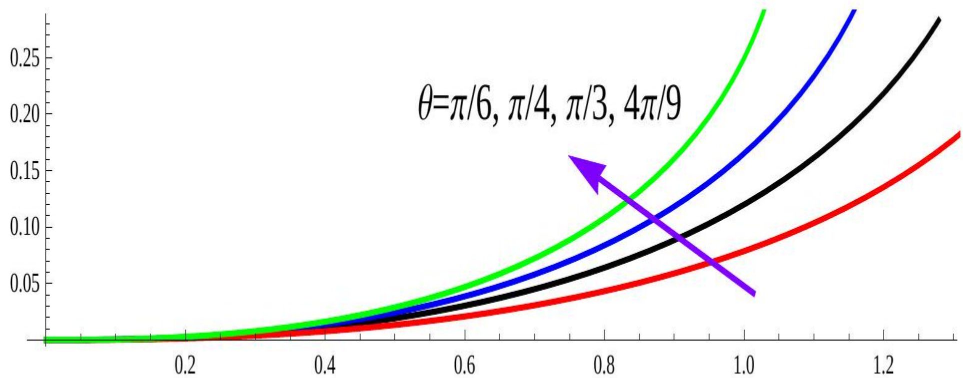
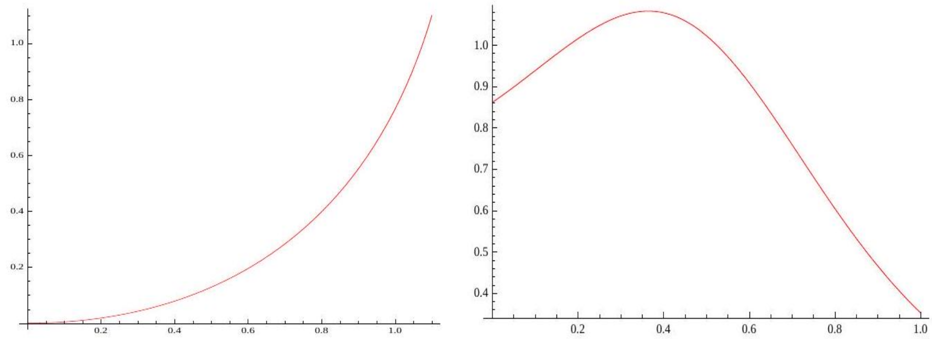


Figure 8. Ball spiral curves with different angles



(a) Cubic ball curve (b) Cubic ball curvature

Figure 9. Cubic ball curve with curvature graph

6. CONCLUSION

In this paper, a cubic Ball spiral segment with free parameters was developed. It was identified that the curvature of the spiral curve is always monotone. Whereas, the point of zero curvature introduces flat spots at points where they may not be desirable. Using proposed cubic Ball spiral curve, such flat spots can be avoided. We also compared the cubic Ball curve with the cubic Ball spiral curve. Although, the cubic Ball curve may or may not be fair; however, the cubic Ball spiral is always a fair curve. Similarly, the cubic Ball curve may or may not be monotone, although the curvature of cubic Ball spiral curve is always monotone.

Conflict of Interests

The authors declare no conflict of interest.

Acknowledgments

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