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Three-Player Gambler's Ruin Problem: Some Extensions

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Abstract

For calculating the expected ruin time of the classic three-player symmetric game, Sandell derived a general formula by introducing an appropriate martingale and stopping time. However, the martingale approach is not appropriate to determine the ruin time of asymmetric game. In general, ruin probabilities in both cases, that is, symmetric and asymmetric games as well as the expected ruin time for the asymmetric games are still need to be calculated. The current work is also about three-player gambler's ruin problem with some extensions. We provide expressions for the ruin time with or without ties when all the players have equal or unequal initial fortunes. Finally, the validity of the asymmetric game is also tested through a Monte Carlo simulation study.

Keywords: ruin time, symmetric and asymmetric games, three-player game, ties, unequal initial fortune

Introduction

In the classical three-player gambler's ruin problem, three players A_1, A_2 and A_3 start a game with arbitrary initial stakes, say a_1, a_2 and a_3 dollars, respectively. In each round, based on randomness, the winner (with probability p_i, where i = 1, 2, 3 and $\sum p_i = 1$) receives one dollar from both the losing players. The game continues until one of them is left with zero dollars. Of interest is the ruin probability for each player and the ruin time of the game. To date, no corresponding expressions for ruin probabilities have been found for both types of games, that is, symmetric and asymmetric games. Ruin time was claimed as an unsolved problem in the early 1960s, see for example [1]. However, in the late 1980s, Sandell [2] accepted this challenge and proposed the following formula with the help of martingale and the optional stopping theorem:

$$E(T)=\frac{a_1a_2a_3}{a_1+a_2+a_3-2} \tag{1}$$

where T denotes the first ruin time. The expression given in equation (1) is valid for both cases, either $a_1=a_2=a_3$ or $a_1\neq a_2\neq a_3$. The literature on asymmetric game for more than two players is more limited and no general expression is available for the ruin time to date. In this regard, [3, 4] provided five different expressions to solve this type of game up to six dollars each as follows:

$$E(T)=\begin{cases} c, & for\ 1\le c\le 2; \\ \frac{3}{1-\alpha}, & for\ c=3; \\ 1+\frac{3}{1-2\alpha}, & for\ c=4; \\ 2+3\left(1+\frac{20\alpha(3-\alpha+\beta)}{4!\left(1-3\alpha+\frac{20\alpha^2}{4!}\right)}\right), & for\ c=5 \end{cases} \tag{2}$$

In equation (2), $\alpha=3!\,p_1p_2p_3$, $\beta=p_1^2+p_2^2+p_3^2$ and $c:=a_1=a_2=a_3$. For c = 6, their given expressions including matrix and vector are very difficult to solve without a software. Recently, [5] provided a single approximation as an alternative of the five separate expressions of [3, 4]. Their proposed structure is stated in the following form:

$$E(T)=1+\frac{\xi_1}{\xi_1+\xi_2+\xi_3}, \tag{3}$$

Where,

$$\xi_1=(a_1+2)(a_2-1)(a_3-1)p_1+(a_1-1)(a_2+2)(a_3-1)p_2+(a_1-1)(a_2-1)(a_3+2)p_3$$

$$\xi_2=(a_1+4)(a_2-2)(a_3-2)p_1^2+(a_1-2)(a_2+4)(a_3-2)p_2^2+(a_1-2)(a_2-2)(a_3+4)p_3^2$$

$$\xi_3 = (a_1+1)(a_2+1)(a_3-2)\,p_1p_2 + (a_1+1)(a_2-2)(a_3+1)\,p_1p_3 + (a_1-2)(a_2+1)(a_3+1)\,p_2p_3$$

In this research, we adapt a direct way to derive the ruin time for the three-player game with unequal initial fortunes. Also, we extend the symmetric game with the involvement of ties in each round of the game. In Section 2, we delineate the mathematical derivations for both the symmetric and asymmetric games with and without ties. Section 3 evaluates the results of the proposed asymmetric game with a Monte Carlo simulation study. At the end, a brief summary with observations is discussed in Section 4.

2. Mathematical Description

Let us assume that waiting time for the gambler's ruin game is finite. First, we consider the ruin time for two-player game, conditional expectation and random variable (see for example [6, 7]), in the following way:

$$E(T) = \sum_{n=0}^{\infty} nP(n|a_1) = \sum_{n=0}^{\infty} nP(n|a_2) = T_{a_1,a_2}$$

where n signifies the remaining steps needed to complete the game, whereas a_1 and a_2 signify the initial fortunes for players A_1 and A_2, respectively. Moreover,

$$P(n|a_1) = pP(n-1|a_1+1) + (1-p)P(n-1|a_1-1) = P(n|a_2)$$

where p and $1-p$ represent the winning probabilities for players A_1 and A_2, respectively. Then, T_{a_1,a_2} can be computed by using the law of total expectation as follows:

$$T_{a_1,a_2} = 1 + pT_{a_1+1,a_2-1} + (1-p)T_{a_1-1,a_2+1} \tag{4}$$

2.1. Ruin Time: Symmetric Case

For the three-player gambler's ruin problem, waiting time is also finite and expected duration can be written as follows:

$$E(T) = T_{a_1,a_2,a_3}$$

In this perspective, one out of the following three events has to occur: either the player A_1 wins and not the other two players, or the winner player is A_2 (whereas players A_1 and A_3 lose), or player A_3 is declared as the winner (players A_1 and A_2 lose). Similarly, the ruin time of the classic three-player game must satisfy the following difference equation:

$$T_{a_1,a_2,a_3} = 1 + pT_{a_1+2,a_2-1,a_3-1} + pT_{a_1-1,a_2+2,a_3-1} + pT_{a_1-1,a_2-1,a_3+2} \tag{5}$$

where $p = \dfrac{1}{3}$. We restrict this game up to the stage where at least one of the three players suffers a complete loss with the following boundary conditions:

$$T_{0,a_2,a_3} = T_{a_1,0,a_3} = T_{a_1,a_2,0} = T_{0,0,a_3} = T_{0,a_2,0} = T_{a_1,0,0} := 0 \tag{6}$$

System (5) is an inhomogeneous difference equation and should have two solutions, that is, the complementary and the particular solution. Then, the general solution takes the following form:

$$T_{a_1,a_2,a_3} = T^c_{a_1,a_2,a_3} + T^p_{a_1,a_2,a_3} \tag{7}$$

where $T^c_{a_1,a_2,a_3}$ agrees with the homogeneous part (without constant 1) and $T^p_{a_1,a_2,a_3}$ is the particular solution.

Based on the non-zero determinant, we determined the value of $T^c_{a_1,a_2,a_3}$ as a complementary solution of system (5) and found that it was zero. For illustration, if there are arbitrary individual fortunes involved in the game, such as $a_1 = 2$, $a_2 = 3$ and $a_3 = 4$, then system (5) can be written without constant 1 as follows:

$$\left.\begin{aligned}
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{4,2,3} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},\\
T_{2,3,4} &= pT_{4.2.3} + pT_{1,5,3} + pT_{1,2,6},
\end{aligned}\right\} \tag{8}$$

In this system, there are 9 equations for 9 unknowns. Using the boundary conditions mentioned in equation (6) and by solving it simultaneously we get $T_{2,3,4} = 0$. Hence, all other unknowns are also equal to zero and it proves that the homogeneous part (without 1) of system (5) with any arbitrary values of a_1, a_2 and a_3 is always given a zero value.

For the particular solution of system (5), we can take the lowest order of the third-order polynomial with a_1, a_2 and a_3. Hence, the value of $T^p_{a_1,a_2,a_3}$ is presented in the following form:

$$T^p_{a_1,a_2,a_3} = C(a_1 a_2 a_3)$$

where C is an arbitrary constant. We obtained the value of C using system (5) as follows:

$$C = \frac{1}{a_1 + a_2 + a_3 - 2}$$

and the particular solution can be generated as follows:

$$T^p_{a,b,c} = \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3 - 2}$$

Hence, the general solution is in the following form:

$$E(T) = T^c_{a,b,c} + T^p_{a,b,c} = \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3 - 2}$$

2.2. Ruin Time with Ties: Symmetric Case

By involving the ties in each round of the three-player game, system (5) can be written as follows:

$$T_{a_1,a_2,a_3} = 1 + pT_{a_1+2,a_2-1,a_3-1} + pT_{a_1-1,a_2+2,a_3-1} + pT_{a_1-1,a_2-1,a_3+2} + (1-3p)T_{a_1,a_2,a_3}$$

It can be written in a more simple form as follows:

$$T_{a_1,a_2,a_3} = \frac{1}{3p}(1 + pT_{a_1+2,a_2-1,a_3-1} + pT_{a_1-1,a_2+2,a_3-1} + pT_{a_1-1,a_2-1,a_3+2}) \qquad (9)$$

Complementary solution of system (9) remains zero and the particular solution will be similar as above mentioned ($T^p_{a_1,a_2,a_3} = C(a_1 a_2 a_3)$). We obtain constant C using equation (9) as $C = \dfrac{1}{3p(a_1 + a_2 + a_3 - 2)}$ and the particular solution is given in the following form:

$$T^{p}_{a,b,c} = \frac{a_1 a_2 a_3}{3p(a_1 + a_2 + a_3 - 2)}$$

Hence, the general solution, when ties are involved in the game, takes the following form:

$$E(T) = T^{c}_{a,b,c} + T^{p}_{a,b,c} = \frac{a_1 a_2 a_3}{3p(a_1 + a_2 + a_3 - 2)}$$

2.3. Ruin Time: Asymmetric Case

For the asymmetric game, ruin time of the classic three-player game must satisfy the following difference equation:

$$T_{a_1,a_2,a_3} = 1 + p_1 T_{a_1+2,a_2-1,a_3-1} + p_2 T_{a_1-1,a_2+2,a_3-1} + p_3 T_{a_1-1,a_2-1,a_3+2} \quad (10)$$

System (10) also depends on inhomogeneous difference equations and should have two solutions, that is, the complementary and the particular solution. Unequal probability does not affect the complementary solution and, as usual, it comes as zero.

For the particular solution of system (10), we can take the lowest polynomial with order three because of the availability of the complementary solution. So, the particular solution

$T^{p}_{a_1,a_2,a_3}$ takes the following form:

$$T^{p}_{a_1,a_2,a_3} = C(a_1 a_2 a_3)$$

where C is an arbitrary constant. We obtain constant C using the system (10) as follows:

$$C = \frac{1}{a_1 a_2 a_3 - p_1 \Pi_1 - p_2 \Pi_2 - p_3 \Pi_3}$$

where

$$\Pi_1 = (a_1 + 2)(a_2 - 1)(a_3 - 1)$$
$$\Pi_2 = (a_1 - 1)(a_2 + 2)(a_3 - 1)$$

and

$$\Pi_3 = (a_1 - 1)(a_2 - 1)(a_3 + 2)$$

Now, the particular solution can be generated as follows:

$$T^{p}_{a,b,c} = \frac{a_1 a_2 a_3}{a_1 a_2 a_3 - p_1 \Pi_1 - p_2 \Pi_2 - p_3 \Pi_3}$$

Hence, the general solution takes the following form:

$$E(T) = \frac{a_1 a_2 a_3}{a_1 a_2 a_3 - p_1\Pi_1 - p_2\Pi_2 - p_3\Pi_3} \tag{11}$$

2.4. Ruin Time with Ties: Asymmetric Case

By involving the ties in each round of the asymmetric three-player game, system (10) can be written as follows:

$$T_{a_1,a_2,a_3} = 1 + p_1 T_{a_1+2,a_2-1,a_3-1} + p_2 T_{a_1-1,a_2+2,a_3-1} + p_3 T_{a_1-1,a_2-1,a_3+2} + (1 - p_1 - p_2 - p_3) T_{a_1,a_2,a_3}$$

It can be written in a more simple form as follows:

$$T_{a_1,a_2,a_3} = \frac{1}{\sum_{i=1}^{3} p_i}\left(1 + p_1 T_{a_1+2,a_2-1,a_3-1} + p_2 T_{a_1-1,a_2+2,a_3-1} + p_3 T_{a_1-1,a_2-1,a_3+2}\right) \tag{12}$$

Complementary solution of system (12) is still zero and we draw the particular solution as $T^p_{a_1,a_2,a_3} = C(a_1 a_2 a_3)$. Constant C can be determined by using equation (12) as follows:

$$C = \frac{1}{\left(\sum_{i=1}^{3} p_i\right)(a_1 a_2 a_3) - p_1\Pi_1 - p_2\Pi_2 - p_3\Pi_3}$$

The particular solution can be written as follows:

$$T^p_{a,b,c} = \frac{a_1 a_2 a_3}{\left(\sum_{i=1}^{3} p_i\right)(a_1 a_2 a_3) - p_1\Pi_1 - p_2\Pi_2 - p_3\Pi_3}.$$

Hence, the general solution takes the following form:

$$E(T) = \frac{a_1 a_2 a_3}{\left(\sum_{i=1}^{3} p_i\right)(a_1 a_2 a_3) - p_1\Pi_1 - p_2\Pi_2 - p_3\Pi_3} \tag{13}$$

In the next section, we demonstrate our proposition through numerical findings and also compare them with a Monte Carlo simulation study.

3. Numerical Results with Monte Carlo Simulation Study

In this section, we examine the performance of ruin time for unequal initial fortunes with various settings of probabilities. In order to evaluate the performance of the proposed asymmetric game, a Monte Carlo simulation study was conducted through MATLAB programming language. In Table 1, we present two types of results of the expected duration, that is, the pre-decided probability for each player and the assigned probabilities with respect to their initial fortunes (probability proportion to size). One may appreciate the vividly close match between the proposed formula and simulation results, especially in the second case, when the probability of each player is based on their own initial fortunes (a real motive to start the game). At the end, two random clicks with different parametric settings are displayed in Figure 1.

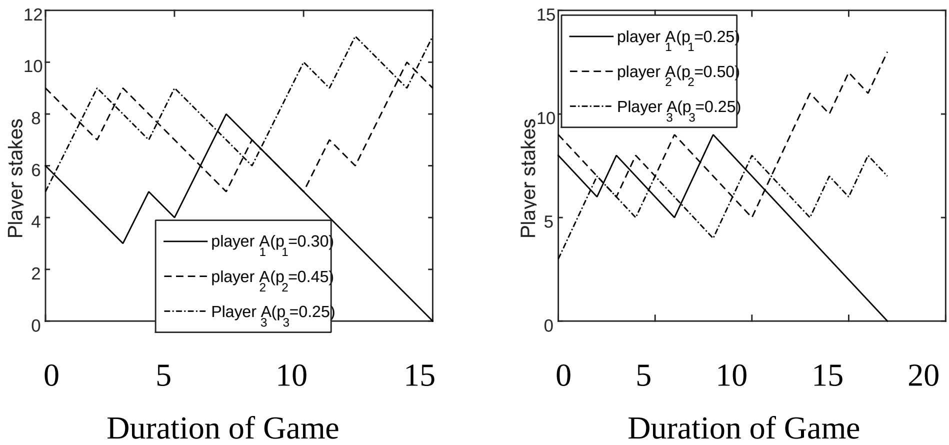

Figure 1. Random clicks of ruin time for various initial stakes with a sample of probabilities

Table 1. Ruin Time for Asymmetric Game: Results of Expression (9) along with 20,000 Simulations

(a_1, a_2, a_3)	(p_1, p_2, p_3)	$E(T)$	**Simulation**
(5,4,1)		1.56	1.8197
(2,6,2)		2.73	2.6681
(5,3,2)	$(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$	3.00	3.1290
(4,4,2)		3.08	3.2460
(3,4,3)		4.29	4.1350

(a_1,a_2,a_3)	(p_1,p_2,p_3)	$E(T)$	**Simulation**
(5,4,1)	$(\frac{a_1}{s},\frac{a_2}{s},\frac{a_3}{s})$	1.22	1.3514
(2,6,2)		2.14	2.3034
(5,3,2)		2.83	2.9699
(4,4,2)		3.08	3.2343
(3,4,3)		4.29	4.3189
(4,8,3)	$(\frac{3}{10},\frac{4}{10},\frac{3}{10})$	6.27	6.4661
(6,6,3)		7.45	7.7058
(4,7,4)		7.57	7.7788
(7,5,3)		7.61	7.7269
(6,5,4)		9.09	8.9396
(4,8,3)	$(\frac{a_1}{s},\frac{a_2}{s},\frac{a_3}{s})$	4.66	4.7934
(6,6,3)		5.68	5.8812
(4,7,4)		6.75	6.8329
(7,5,3)		5.41	5.6023
(6,5,4)		8.22	8.2860
(8,9,3)	$(\frac{1}{4},\frac{2}{4},\frac{1}{4})$	7.45	7.6994
(6,10,4)		8.42	8.5049
(9,7,4)		11.20	10.2194
(8,8,4)		10.24	9.7724
(6,9,5)		10.38	9.8962
(8,9,3)	$(\frac{a_1}{s},\frac{a_2}{s},\frac{a_3}{s})$	5.31	5.2972
(6,10,4)		7.69	7.6961
(9,7,4)		8.48	8.4600
(8,8,4)		8.77	8.7729
(6,9,5)		11.16	11.0336

Note: $s = a_1 + a_2 + a_3$

4. Conclusion

For the three-player game, an explicit formula for the symmetric case is available in literature [2]. On the other hand, the asymmetric case is still unsolved (no explicit formula is available). Recently, Hussain et al, 2021 solved this game, approximately [5]. In the current research, we have also solved this game with a different approach. The basic advantage of this approach is that it is conceptually simple for the students familiar with difference equations. Also, for the very first time in literature, tie is being introduced in this game for both cases, that is, symmetric and asymmetric. The legitimacy of the novel derivation of ruin time for the three-player gambler's ruin problem has been established mathematically and verified through a Monte Carlo simulation study.

Conflict of Interest

The authors declare no conflict of interest.

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