A three-player gambler’s ruin problem: some extensions

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Abstract

For the expected ruin time of the classic three-player symmetric game, Sandell derived a general formula by introducing an appropriate martingale and stopping time. For the case of asymmetric game, the martingale approach is not valid to determine the ruin time. In general, the ruin probabilities for both cases, i.e. symmetric and asymmetric game and expected ruin time for asymmetric game are still awaiting to be solved for this game. The current work is also about three-player gambler’s ruin problem with some extensions as well. We provide expressions for the ruin time with (without) ties when all the players have equal (unequal) initial fortunes. Finally, the validity of asymmetric game is also tested through a Monte Carlo simulation study.

Keywords: Three-player game; symmetric and asymmetric games; the ruin time; unequal initial fortune; ties.

1 Introduction

In classical three-player gambler’s ruin problem, three players A₁, A₂ and A₃ start a game with arbitrary initial stakes, say a₁, a₂ and a₃ dollars, respectively. At each round, based on randomness, the winner player (with probability pᵢ, where i = 1,2,3 and \( \sum p_i = 1 \)) receives one dollar from each of the two others. The game is continued until at least one of them has
zero dollar left. Of interest is the ruin probability for each player and the ruin time of the game. Till to date, no corresponding expressions for ruin probabilities have been found for both cases, i.e. symmetric and asymmetric games. For the ruin time, it was claimed as an unsolved problem in early 1960s, see for example [1]. But in late 1980s, Sandell [2] accepted this challenge and proposed the following formula with the help of martingale and Optional Stopping theorem:

\[ E(T) = \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3 - 2}, \]  

(1)

where, \( T \) is denoted the first ruin time. The expression given in equation (1) is valid for both cases, either \( a_1 = a_2 = a_3 \) or \( a_1 \neq a_2 \neq a_3 \). The literature on asymmetric game for more than two players is more limited and no general expression available for the ruin time till to date. In this regards, [3, 4] provided five different expressions to solve this game up to six dollars each as:

\[ E(T) = \begin{cases} 
  c, & \text{for } 1 \leq c \leq 2; \\
  \frac{3}{1-\alpha}, & \text{for } c = 3; \\
  1 + \frac{3}{1-2\alpha}, & \text{for } c = 4; \\
  2 + 3 \left( 1 + \frac{20\alpha(3-\alpha+\beta)}{4!(1-3\alpha+20\alpha^2/4!)} \right), & \text{for } c = 5.
\]  

(2)

In equation (2), \( \alpha = 3! p_1 p_2 p_3, \beta = p_1^2 + p_2^2 + p_3^2 \) and \( c := a_1 = a_2 = a_3 \). For \( c = 6 \), their given expressions like matrix and vector are very difficult to solve without a software. Recently, [5] have provided a single approximation which is an alternative of the five separate expressions of [3, 4]. Their proposed structure is in the following form:

\[ E(T) = 1 + \frac{\xi_1}{\xi_1 + \xi_2 + \xi_3}, \]  

(3)

where,

\[ \xi_1 = (a_1 + 2)(a_2 - 1)(a_3 - 1)p_1 + (a_1 - 1)(a_2 + 2)(a_3 - 1)p_2 + (a_1 - 1)(a_2 - 1)(a_3 + 2)p_3, \]

\[ \xi_2 = (a_1 + 4)(a_2 - 2)(a_3 - 2)p_1^2 + (a_1 - 2)(a_2 + 4)(a_3 - 2)p_2^2 + (a_1 - 2)(a_2 - 2)(a_3 + 4)p_3^2, \]

\[ \xi_3 = (a_1 + 1)(a_2 + 1)(a_3 - 2)p_1 p_2 + (a_1 + 1)(a_2 - 2)(a_3 + 1)p_1 p_3 + (a_1 - 2)(a_2 + 1)(a_3 + 1)p_2 p_3. \]

In this research, we adapt a direct way to derive the ruin time for three-player game with unequal initial fortunes. Also, we extent the symmetric game with the involvement of ties in each round of the game. In Section 2, we delineate the mathematical derivations for the symmetric and asymmetric games with and without ties. The Section 3 evaluated the results of proposed asymmetric game with a Monte Carlo simulation study. In last, a brief summary with observations about this study is discussed in Section 4.
2 The Mathematical Description

Let us assume that, the waiting time for the gambler’s ruin game is finite. First, we consider the ruin time for two-player game, the conditional expectation and random variable, (see for example, [6, 7]) as in the following way:

\[ E(T) = \sum_{n=0}^{\infty} nP(n|a_1) = \sum_{n=0}^{\infty} nP(n|a_2) = T_{a_1,a_2}, \]

where, \( n \) is the remaining steps to complete the game, whereas \( a_1 \) and \( a_2 \) are the initial fortunes for players \( A_1 \) and \( A_2 \), respectively and,

\[ P(n|a_1) = pP(n-1|a_1 + 1) + (1-p)P(n-1|a_1 - 1) = P(n|a_2), \]

where, \( p \) and \( 1-p \) are the winning probabilities for player \( A_1 \) and \( A_2 \), respectively. Then \( T_{a_1,a_2} \) can be computed by using the law of total expectation as:

\[ T_{a_1,a_2} = 1 + pT_{a_1+1,a_2-1} + (1-p)T_{a_1-1,a_2+1}. \] (4)

2.1 The ruin time: symmetric case

For the three-player gambler’s ruin problem, the waiting time is also finite and expected duration can be written as:

\[ E(T) = T_{a_1,a_2,a_3}. \]

In this perspective, one out of three events has to occur: either the player \( A_1 \) wins and not the other two or the winner player is \( A_2 \) (where \( A_1 \) and \( A_3 \) lose) or player \( A_3 \) is declared as winners (and players \( A_1 \) and \( A_2 \) lose). Similarly as equation (4), the ruin time of the classic three-player game must satisfies the following difference equation:

\[ T_{a_1,a_2,a_3} = 1 + pT_{a_1+2,a_2-1,a_3-1} + pT_{a_1-1,a_2+2,a_3-1} + pT_{a_1-1,a_2-1,a_3+2}, \] (5)

where, \( p = \frac{1}{3} \). We restrict this game up to the stage, where at least one of the players suffers a complete loss with the following boundary conditions:

\[ T_{0,a_2,a_3} = T_{a_1,0,a_3} = T_{a_1,a_2,0} = T_{0,0,a_3} = T_{0,a_2,0} = T_{a_1,0,0} = 0. \] (6)

The system (5) is an inhomogeneous difference equation and should have two solutions, i.e. the complementary and the particular. Then the general solution is in the following form:

\[ T_{a_1,a_2,a_3} = T^c_{a_1,a_2,a_3} + T^p_{a_1,a_2,a_3}, \] (7)

where, \( T^c_{a_1,a_2,a_3} \) agrees with the homogeneous part (i.e. without constant 1) and \( T^p_{a_1,a_2,a_3} \) is the particular solution.

Because of the non-zero determinant, we can determine the value of \( T^c_{a_1,a_2,a_3} \), as a complementary solution of the system (5), and found it as zero. For illustration, if there are an arbitrary individual fortunes involved in the game, such as, \( a_1 = 2, a_2 = 3 \) and \( a_3 = 4 \), then the system (5) can be written without constant 1, as:
whereas in this system, there are 9 equations for 9 unknowns. By using the boundary conditions, which are mentioned in equation (6) and then solving it simultaneously, we get, $T_{2,3,4} = 0$. Hence, rest of all other unknowns are also equal to zero and this will prove that the homogeneous part (without 1) of the system (5) with any arbitrary values of $a_1$, $a_2$ and $a_3$, always given a zero value.

For the particular solution of the system (5), we can take the lowest order of third-order polynomial with $a_1$, $a_2$ and $a_3$. Hence, the value of $T_{p, a_1, a_2, a_3}$ is in the following form:

$$T_{p, a_1, a_2, a_3} = C(a_1a_2a_3),$$

where, $C$ is an arbitrary constant. We obtained the value of $C$, by using the system (5) as:

$$C = \frac{1}{a_1 + a_2 + a_3 - 2},$$

and the particular solution can be generated as:

$$T_{p, a_1, a_2, a_3} = \frac{a_1a_2a_3}{a_1 + a_2 + a_3 - 2}.$$

Hence, the general solution is in the following form:

$$E(T) = T_{c, a_1, a_2, a_3} + T_{p, a_1, a_2, a_3} = \frac{a_1a_2a_3}{a_1 + a_2 + a_3 - 2}.$$

### 2.2 The ruin time with ties: symmetric case

By involving the ties in each round of the three-player game, the system (5) can be written as:

$$T_{a_1, a_2, a_3} = 1 + pT_{a_1 + 2, a_2 - 1, a_3 - 1} + pT_{a_1 - 1, a_2 + 2, a_3 - 1} + pT_{a_1 - 1, a_2 - 1, a_3 + 2} + (1 - 3p)T_{a_1, a_2, a_3}.$$

It can be written in more simple form as:

$$T_{a_1, a_2, a_3} = \frac{1}{3p}(1 + pT_{a_1 + 2, a_2 - 1, a_3 - 1} + pT_{a_1 - 1, a_2 + 2, a_3 - 1} + pT_{a_1 - 1, a_2 - 1, a_3 + 2}).$$ (9)
The complementary solution of the system (9) remains zero and the particular solution will be similar as above mentioned (i.e. $T_{p}^{a,b,c} = C(a,a_{1},a_{2})$). We obtain the constant $C$ by using equation (9) as, $C = \frac{1}{3p(a_{1} + a_{2} + a_{3} - 2)}$ and the particular solution is in the following form:

$$T_{a,b,c}^{p} = \frac{a_{1}a_{2}a_{3}}{3p(a_{1} + a_{2} + a_{3} - 2)}.$$

Hence, the general solution, when ties involved in the game, will be in the following form:

$$E(T) = T_{a,b,c}^{c} + T_{a,b,c}^{p} = \frac{a_{1}a_{2}a_{3}}{3p(a_{1} + a_{2} + a_{3} - 2)}.$$

### 2.3 The ruin time: asymmetric case

For the asymmetric game, the ruin time of the classic three-player game must satisfies the following difference equation:

$$T_{a_{1},a_{2},a_{3}} = 1 + p_{1}T_{a_{1}+2,a_{2}-1,a_{3}-1} + p_{2}T_{a_{1}+1,a_{2}+2,a_{3}-1} + p_{3}T_{a_{1}-1,a_{2}+1,a_{3}+2}.$$  \hspace{1cm} (10)

The system (10) is also depends on inhomogeneous difference equations and should have two solutions, i.e. the complementary and the particular. The unequal probabilities does not effect on the complementary solution and as usual it comes as zero.

For the particular solution of the system (10), we can take any lowest polynomial with order three because of no availability of complementary solution. So, the particular solution, $T_{a_{1},a_{2},a_{3}}^{p}$, is in the following form:

$$T_{a_{1},a_{2},a_{3}}^{p} = C(a,a_{2},a_{3}),$$

where, $C$ is an arbitrary constant. We obtain the constant $C$, by using the system (10) as:

$$C = \frac{1}{a_{1}a_{2}a_{3} - p_{1}\Pi_{1} - p_{2}\Pi_{2} - p_{3}\Pi_{3}},$$

where,

$$\Pi_{1} = (a_{1} + 2)(a_{2} - 1)(a_{3} - 1),$$

$$\Pi_{2} = (a_{1} - 1)(a_{2} + 2)(a_{3} - 1),$$

and

$$\Pi_{3} = (a_{1} - 1)(a_{2} - 1)(a_{3} + 2).$$

Now, the particular solution can be generated as:

$$T_{a,b,c}^{p} = \frac{a_{1}a_{2}a_{3}}{a_{1}a_{2}a_{3} - p_{1}\Pi_{1} - p_{2}\Pi_{2} - p_{3}\Pi_{3}}.$$

Hence, the general solution is in the following form:

$$E(T) = \frac{a_{1}a_{2}a_{3}}{a_{1}a_{2}a_{3} - p_{1}\Pi_{1} - p_{2}\Pi_{2} - p_{3}\Pi_{3}}.$$  \hspace{1cm} (11)
2.4 The ruin time with ties: asymmetric case

By involving the ties in each round of the asymmetric three-player game, the system (10) can be written as:

\[ T_{a_1,a_2,a_3} = 1 + p_1 T_{a_1+2,a_2-1,a_3-1} + p_2 T_{a_1-1,a_2+2,a_3-1} + p_3 T_{a_1-1,a_2-1,a_3+2} + (1 - p_1 - p_2 - p_3) T_{a_1,a_2,a_3}. \]

It can be written in more simple form as:

\[ T_{a_1,a_2,a_3} = \frac{1}{3} \left( 1 + p_1 T_{a_1+2,a_2-1,a_3-1} + p_2 T_{a_1-1,a_2+2,a_3-1} + p_3 T_{a_1-1,a_2-1,a_3+2} \right). \]  

(12)

The complementary solution of the system (12) is still zero and we take the particular solution as: \( T_{a_1,a_2,a_3} = C(a_1,a_2,a_3) \). The constant \( C \) can be determined after using equation (12) as:

\[ C = \frac{1}{\sum_{i=1}^{3} p_i (a_1,a_2,a_3) - p_1 \Pi_1 - p_2 \Pi_2 - p_3 \Pi_3} , \]

and the particular solution can be written as:

\[ T_{a,b,c}^p = \frac{a_1 a_2 a_3}{\sum_{i=1}^{3} p_i (a_1,a_2,a_3) - p_1 \Pi_1 - p_2 \Pi_2 - p_3 \Pi_3} . \]

Hence, the general solution is in the following form:

\[ E(T) = \frac{a_1 a_2 a_3}{\sum_{i=1}^{3} p_i (a_1,a_2,a_3) - p_1 \Pi_1 - p_2 \Pi_2 - p_3 \Pi_3} . \]  

(13)

In next section, we will demonstrate our proposition through numerical findings and also compare with a Monte Carlo simulation study.

3 Numerical Results with Monte Carlo Simulation Study

In this section, we examined the performance of the ruin time for unequal initial fortunes with various settings of probabilities. In order to evaluate the performance of the proposed asymmetric game, a Monte Carlo simulation study was conducted through MATLAB programming language. In Table 1, we present two types of results of the expected duration, i.e. the pre-decided probability for each player and assigning probabilities with respect to their initial fortunes (i.e. probability proportion to size). One may appreciate the vividly close match between the proposed formula and simulation results, especially in second case, when probability of each player based on his/her own initial fortunes (i.e. a real motive to start the game). At the end, two random clicks with different parametric settings are displayed in Figure 1.
4 Conclusion

For three player game, an explicit formula for symmetric case is only available in literature (see for example, [2]). On the other hand, the game for asymmetric case is still awaiting to be solved (no explicit formula is available). Recently, [5] solved this game approximately. In current research, we also solve this game with a different approach. The basic advantage of this approach is, that it is conceptually simple for the student familiar with difference equations. Also, very first time in literature, the tie is being introduced in this game for both cases, i.e. symmetric and asymmetric. The legitimacy of the novel derivation of the ruin time for the three-player gambler’s ruin problem is established mathematically and verified through a Monte Carlo simulation study.

Disclosure Statement

No conflict of interest is declared by authors.

References


Table 1: The ruin time for the case of asymmetric game: the results of expression (9), along with 20,000 simulations.

<table>
<thead>
<tr>
<th>$(a_1, a_2, a_3)$</th>
<th>$(p_1, p_2, p_3)$</th>
<th>$E(T)$</th>
<th>Simulation</th>
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Note: $s = a_1 + a_2 + a_3$
Figure 1: Some random clicks of ruin time for various initial stake with a sample of probabilities.