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Title: **Operational Matrix of Fractional Order Integration and Its Application to Solve Fractional Differential Equations (FDEs) Using Haar Wavelet Collocation Method (HWCM)**

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Operational Matrix of Fractional Order Integration and Its Application to Solve Fractional Differential Equations (FDEs) using Haar Wavelet Collocation Method (HWCM)

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Abstract

Wavelets play an essential part in numerical analysis. In this study, a novel numerical technique to solve fractional differential equations (FDEs) corresponding to initial conditions is presented using Haar wavelet approximations. Haar wavelet is first presented with an operational matrix of fractional order integration. Then, illustrative examples are presented to signify the validity and applicability of the proposed method.

Keywords: fractional differential equations (FDEs), fractional order integration, Haar wavelet collocation method (HWCM), operational matrix.

Introduction

Fractional differential equations (FDEs) are overviews of ordinary differential equations to an arbitrary (non-integer) order. FDEs have attracted significant attention because of their ability to model composite phenomena, such as visco-elastic materials [1], economics [2], continuum and statistical mechanics [3], solid mechanics [4], and many more. Due to extensive application of FDEs in science and engineering, considerable research has been conducted in this area. Hence, significant efforts to develop numerical techniques to solve FDEs are currently underway. Some of these include Fourier transforms [5], Laplace transforms [6], Adomian decomposition method [7], Variational iteration method [8], and Homotopy analysis method [9]. Further research conducted on FDEs is reflected in [10-15].

Orthogonal functions have been used when dealing with various problems of dynamical systems. The approach deployed while using orthogonal functions is to convert the underlying differential equation into an integral equation through integration, approximating various signals involved in the equation using truncated orthogonal functions, and using the operational matrix of integration to eliminate integral operations. This matrix can be uniquely determined based on the particular orthogonal functions performed using Haar wavelet.

Wavelet theory is a relatively novel area in mathematical research. It is useful in a wide range of science and engineering disciplines. Wavelets are effectively applied in signal analysis for creating waveform representations and segmentations, time–frequency analysis, and fast algorithms for easy implementation [16]. Wavelets allow the accurate representation of a variety of functions and operators [17, 18]. Furthermore, they establish a connection of wavelets with fast numerical algorithms [19].

In the current study, we present Haar wavelet collocation method (HWCM) for the solution of multi-term linear and nonlinear FDEs. Wavelets have been in use since 1980 to solve ordinary differential equations (ODEs) and partial differential equations (PDEs). Wavelet algorithms used to solve ODEs and PDEs are based on Galerkin techniques or on the collocation method. Evidently, attempts to simplify wavelet solutions for ODEs and PDEs are in vogue. One possibility is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order one) consist of piecewise constant functions and are, therefore, the simplest orthonormal wavelets with a compact support. Recently, the Haar wavelet method was applied to solve some ODEs and PDEs by selected researchers [20-27]. HWCM [28] was used to solve multi-term FDEs. In this study, we apply HWCM to solve specific classes of linear and nonlinear FDEs and obtain solutions for different fractional orders.

The outline of this paper is as follows: Haar wavelets and their operational matrix of integration of fractional order are presented in Section 2. The method of solution is discussed in Section 3. Numerical solutions with error analysis of the given problems are given in Section 4. Lastly, Section 5 deals with the conclusion drawn by applying the proposed technique.

2. Haar Wavelets

We use the simplest wavelet function known as the Haar wavelet. The scaling function $H_1(t)$ for the family of Haar wavelets is defined as

$$H_1(t)=\begin{cases}1 & for \quad t\in[0,1),\\ 0 & Otherwise.\end{cases} \tag{2.1}$$

The Haar wavelet family for $t\in[0,1)$ is defined as

$$H_i(t)=\begin{cases}1 & for \quad t\in\left[\frac{k}{m},\frac{k+0.5}{m}\right),\\ -1 & for \quad t\in\left[\frac{k+0.5}{m},\frac{k+1}{m}\right),\\ 0 & Otherwise.\end{cases} \tag{2.2}$$

Here, $m=2^l$, $l=0,1,\ldots,J$, is the resolution level of the wavelet and integer $k=0,1,\ldots,m-1$ that denotes the translation parameter. The level of maximum resolution is J. The index i in the left-hand side (LHS) of Eq. (2.2) is measured by $i=m+k+1$. In case of minimal values $m=1,\ k=0$ then $i=2$. The maximum value of i is $N=2^{J+1}$. Now, we define the collocation points $t_j=\frac{j-0.5}{N}, j=1,2,\ldots,N$ and discretize the Haar function $H_i(t)$. In this way, we get Haar matrix $\mathrm{H}(i,j)=H_i(t_j)$ with the dimension $N\times N$.

If $J=2\Rightarrow N=8$, then using Eq. (2.2) we have Haar matrix

$$\mathrm{H}(8,8)=\begin{pmatrix}1&1&1&1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&0&0&0&0\\0&0&0&0&1&1&-1&-1\\1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0\\0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&1&-1\end{pmatrix}.$$

2.1. Operational Matrix of Integration of Fractional Order

Now, we establish an operational matrix of integration using Haar wavelets. The operational matrix F_α of integration of fractional order α using Eq. (2.2) is given by

$$F_{\alpha,i}(t)=\begin{cases} f_1 & for \quad t\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right), \\ f_2 & for \quad t\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right), \\ f_3 & for \quad t\in\left[\dfrac{k+1}{m},1\right), \\ 0 & Otherwise, \end{cases} \tag{2.3}$$

where

$$f_1=\frac{1}{\Gamma(\alpha+1)}\left(t-\frac{k}{m}\right)^{\alpha}, \quad f_2=\frac{1}{\Gamma(\alpha+1)}\left\{\left(t-\frac{k}{m}\right)^{\alpha}-2\left(t-\frac{k+0.5}{m}\right)^{\alpha}\right\}$$

and

$$f_3=\frac{1}{\Gamma(\alpha+1)}\left\{\left(t-\frac{k}{m}\right)^{\alpha}-2\left(t-\frac{k+0.5}{m}\right)^{\alpha}+\left(t-\frac{k+1}{m}\right)^{\alpha}\right\}.$$

The integral matrix F_α has the elements $F_{\alpha,i}(t)=F_\alpha(i,j)$, if $J=2\Rightarrow N=8$ from Eq. (2.3).

For instance, $\alpha=1/2$, we have

$$F_{\frac{1}{2},i}(t)=F_{\frac{1}{2}}(8,8)=\begin{pmatrix} 0.2821 & 0.4886 & 0.6308 & 0.7464 & 0.8463 & 0.9356 & 1.0171 & 1.0925 \\ 0.2821 & 0.4886 & 0.6308 & 0.7464 & 0.2821 & -0.0416 & -0.2445 & -0.4002 \\ 0.2821 & 0.4886 & 0.0666 & -0.2309 & -0.1332 & -0.0685 & -0.0447 & -0.0323 \\ 0 & 0 & 0 & 0 & 0.2821 & 0.4886 & 0.0666 & -0.2309 \\ 0.2821 & -0.0756 & -0.0643 & -0.0266 & -0.0156 & -0.0106 & -0.0078 & -0.0061 \\ 0 & 0 & 0.2821 & -0.0756 & -0.0643 & -0.0266 & -0.0156 & -0.0106 \\ 0 & 0 & 0 & 0 & 0.2821 & -0.0756 & -0.0643 & -0.0266 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2821 & -0.0756 \end{pmatrix}$$

and for $\alpha=3/2$, we have

$$F_{3/2,i}(t)=F_{3/2}(8,8)=\begin{pmatrix} 0.0118 & 0.0611 & 0.1314 & 0.2177 & 0.3174 & 0.4288 & 0.5509 & 0.6828 \\ 0.0118 & 0.0611 & 0.1314 & 0.2177 & 0.2938 & 0.3067 & 0.2881 & 0.2475 \\ 0.0118 & 0.0611 & 0.1079 & 0.0955 & 0.0663 & 0.0545 & 0.0476 & 0.0429 \\ 0 & 0 & 0 & 0 & 0.0118 & 0.0611 & 0.1079 & 0.0955 \\ 0.0118 & 0.0376 & 0.0210 & 0.0159 & 0.0134 & 0.0118 & 0.0107 & 0.0098 \\ 0 & 0 & 0.0118 & 0.0376 & 0.0210 & 0.0159 & 0.0134 & 0.0118 \\ 0 & 0 & 0 & 0 & 0.0118 & 0.0376 & 0.0210 & 0.0159 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0118 & 0.0376 \end{pmatrix}$$

3. Method of Solution

In this section, we discuss the method of solution for the proposed technique. Consider the linear or nonlinear FDEs of the form

$$D^{\alpha_1}u(t)=f\left(t,u,u^2,D^{\alpha_2}u,\ldots\right),\ 1<\alpha_1\le 2,\ 0<\alpha_2\le 1 \qquad (3.1)$$

with initial conditions $u(0)=a,\ u'(0)=b$, where $f\left(t,u,u^2,D^{\alpha_2}u,\ldots\right)$ is a function of independent or dependent variables or constants. The solution $u(t)$ of the Eq. (3.1) can be obtained using the following procedure.

Step 1: Assume that

$$D^{\alpha_1}u(t)=\sum_{i=1}^{N}c_iH_i(t), \qquad (3.2)$$

where c_i 's, $i=1,2,\ldots,N$ are Haar coefficients to be calculated.

Step 2: By integrating Eq. (3.2) and using initial conditions, we have

$$D^{\alpha_2}u(t)=b+\sum_{i=1}^{N}c_iF_{\alpha_2,i}(t) \qquad (3.3)$$

and again, by integrating Eq. (3.3) with the condition, we get

$$u(t)=a+bt+\sum_{i=1}^{N}c_iF_{\alpha_1,i}(t). \qquad (3.4)$$

Step 3: Substituting Eq. (3.2) with Eq. (3.4) in Eq. (3.1), we get the following system of equations.

$$\sum_{i=1}^{N} c_i H_i(t) = f\left(t, a+bt+\sum_{i=1}^{N} c_i F_{\alpha_1,i}(t), \left(a+bt+\sum_{i=1}^{N} c_i F_{\alpha_1,i}(t)\right)^2, b+\sum_{i=1}^{N} c_i F_{\alpha_2,i}(t), \ldots\right). \quad (3.5)$$

Step 4: Solving Eq. (3.5) using MATLAB, we obtain the Haar wavelet coefficients c_i's. On substituting the values of c_i's in Eq. (3.4), we get the desired HWCM-based numerical solution of the problem given in Eq. (3.1). The error can be calculated by using $E = |u_e - u_a|$ and $E_{\max} = \max|u_e - u_a|$, where $u_e \,\&\, u_a$ are exact and approximate solutions, respectively.

4. Numerical Examples

In this section, we apply HWCM to solve some classes of linear and nonlinear FDEs, as discussed in Section 3, using the following examples.

Example 1. Consider the FDE

$$D^2u(t) + 3Du(t) + 2D^{\alpha_1}u(t) + D^{\alpha_2}u(t) + 5u(t) = f(t), \quad 0 < \alpha_2 < \alpha_1 < 1 \quad (4.1)$$

with the initial conditions $u(0)=1, \; u'(0)=0$, where

$f(t) = 1 + 3t + \frac{2}{\Gamma(3-\alpha_1)} t^{2-\alpha_1} + \frac{1}{\Gamma(3-\alpha_2)} t^{2-\alpha_2} + 5\left(1+\frac{t^2}{2}\right)$ and

$\alpha_2 = 0.0159, \; \alpha_1 = 0.1379$.

As per the method explained in Section 3, we obtain the desired HWCM solution $u(t)$ of Eq. (4.1). It agrees with the exact solution $u(t) = 1 + \frac{t^2}{2}$ by increasing the collocation points and is presented in Fig. 1. Error analysis is given in Table 1 and also depicted via Figure 2.

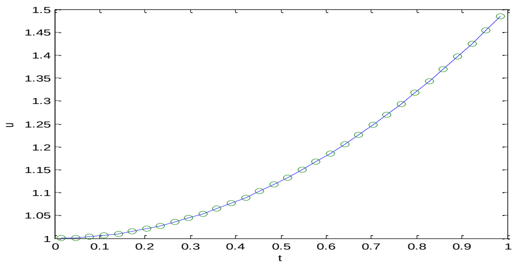

Figure 1. Comparison of HWCM solution with the exact solution for $N = 32$ (Example 1).

Table 1. Error analysis of HWCM (Example 1)

N	$E_{\max} = \max\lvert u_e - u_a \rvert$
16	1.3314e-08
32	2.1100e-08
64	7.3833e-08
128	2.4203e-14
256	8.9262e-14

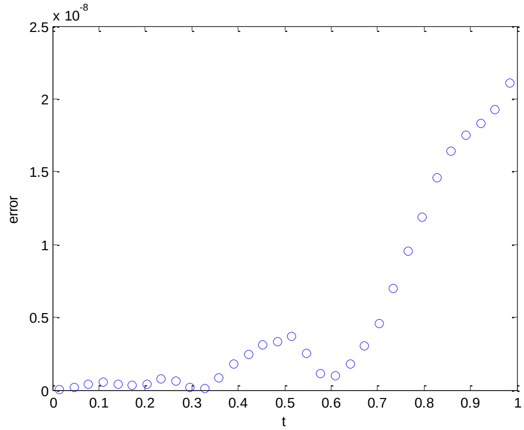

Figure 2. Error analysis (Example 1).

Example 2. Now, consider another FDE

$$D^{\alpha_1}u(t) = D^{\alpha_2}u(t) + f(t) \ , \ 1 < \alpha_1 \le 2, \ 0 < \alpha_2 \le 1 \tag{4.2}$$

with the initial conditions $u(0) = 0, \ u'(0) = 1 - e^{-1}$, where $f(t) = -e^{t-1} - 1$.

As per the procedure explained in Section 3, we obtain the desired HWCM solution $u(t)$ of Eq. (4.2) for different values of α_1. It agrees with the exact solution $u(t) = t(1 - e^{t-1})$, when $\alpha_1 = 2$ and is presented in Fig. 3. Error analysis is given in Table 2 and also depicted via Fig. 4 for different values of α_1.

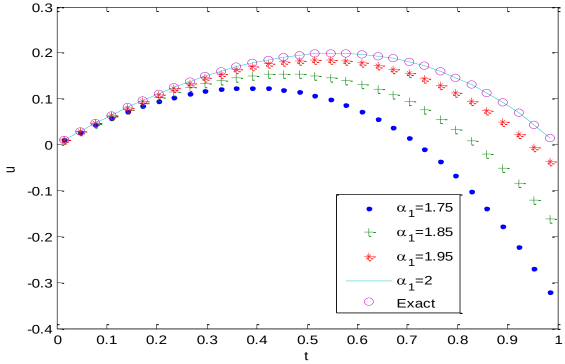

Figure 3. Comparison of HWCM solution for different values of α_1 with exact solution for $N = 32$ (Example 2).

Table 2. Error analysis of HWCM for different values of α_1 (Example 2)

$E_{\max} = \max\lvert u_e - u_a\rvert$				
N	$\alpha_1 = 1.75$	$\alpha_1 = 1.85$	$\alpha_1 = 1.95$	$\alpha_1 = 2$
16	3.2739e-01	1.7263e-01	5.1858e-02	1.0113e-03
32	3.3675e-01	1.7737e-01	5.2776e-02	2.6116e-04
64	3.4202e-01	1.8011e-01	5.3468e-02	6.6365e-05
128	3.4482e-01	1.8159e-01	5.3877e-02	1.6728e-05
256	3.4626e-01	1.8235e-01	5.4098e-02	4.1993e-06

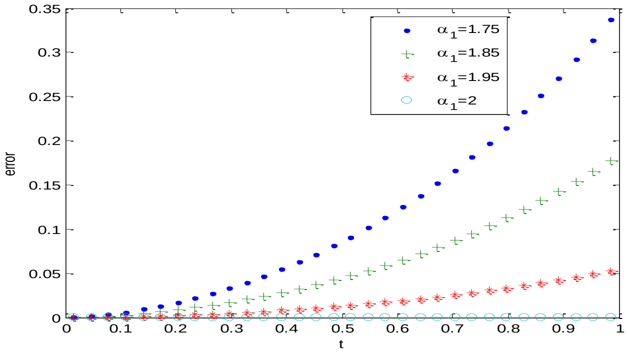

Figure 4. Error analysis for different values of α_1 (Example 2).

Example 3. Now, consider nonlinear FDE

$$D^{\alpha}u(t) = -u^2(t) + 1,\ 0 < \alpha \le 1 \tag{4.3}$$

with the initial condition $u(0) = 0$.

In line with the previous examples, we obtain the desired HWCM solution $u(t)$ of Eq. (4.3) for different values of α. It agrees with the exact solution $u(t) = \dfrac{e^{2t} - 1}{e^{2t} + 1}$, when $\alpha = 1$ and is presented in Fig. 5. Error analysis is given in Table 3 and also depicted via Fig. 6 for different values of α.

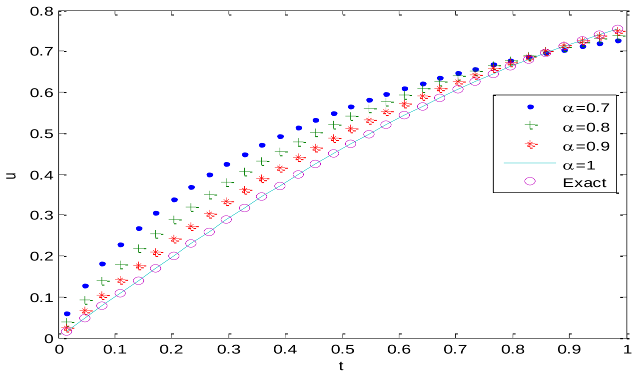

Figure 5. Comparison of HWCM solution for different values of α with exact solution for $N = 32$ (Example 3)

Table 3. Error analysis of HWCM for different values of α (Example 3)

	$E_{\max} = \max\lvert u_e - u_a\rvert$			
N	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
16	1.3854e-01	9.0638e-02	4.4261e-02	1.9711e-04
32	1.3914e-01	9.0943e-02	4.4575e-02	4.9411e-05
64	1.3918e-01	9.1016e-02	4.4641e-02	1.2361e-05
128	1.3925e-01	9.1040e-02	4.4652e-02	3.0908e-06
256	1.3927e-01	9.1045e-02	4.4657e-02	7.7272e-07

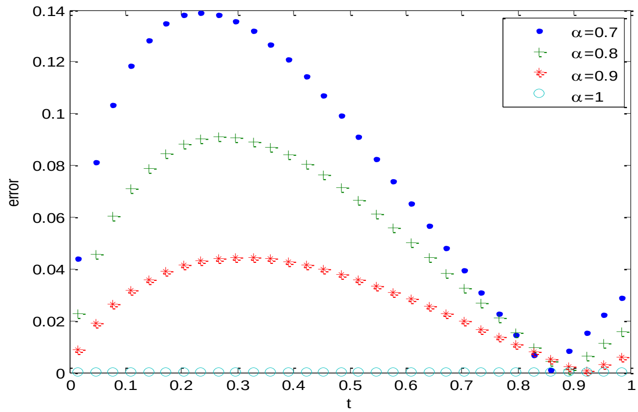

Figure 6. Error analysis for different values of α (Example 3)

Example 4. Lastly, consider another nonlinear FDE

$$D^{\alpha}u(t)=u^2(t)+2\pi^2\cos(2\pi t)-\sin^4(\pi t),\ 1<\alpha\le 2 \tag{4.4}$$

with the initial conditions $u(0)=0,\ u'(0)=0$.

Similar to the previous examples, we obtain the desired HWCM solution $u(t)$ of Eq. (4.4) for different values of α. It agrees with the exact solution $u(t)=\sin^2(\pi t)$, when $\alpha=2$ and is presented in Fig. 7. Error analysis is given in Table 4 and also depicted via Fig. 8 for different values of α.

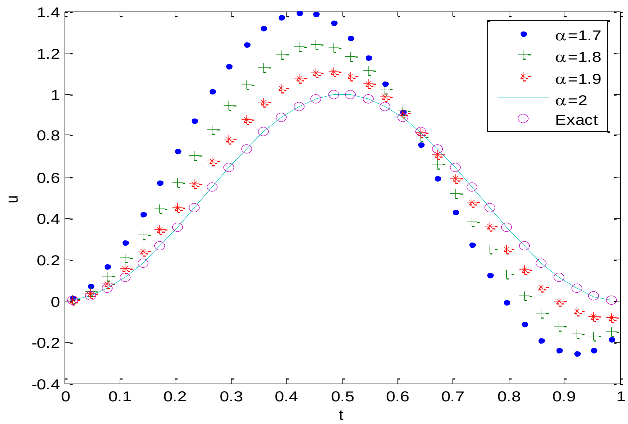

Figure 7. Comparison of HWCM solution for different values of α with exact solution for $N=32$ (Example 4)

Table 4. Error analysis of HWCM for different values of α (Example 4)

	$E_{\max} = \max\lvert u_e - u_a \rvert$			
N	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2$
16	4.9948e-01	3.0727e-01	1.4018e-01	6.7256e-03
32	5.0470e-01	3.1240e-01	1.4524e-01	1.6823e-03
64	5.0736e-01	3.1419e-01	1.4629e-01	4.2073e-04
128	5.0795e-01	3.1451e-01	1.4669e-01	1.0518e-04
256	5.0806e-01	3.1462e-01	1.4678e-01	2.6296e-05

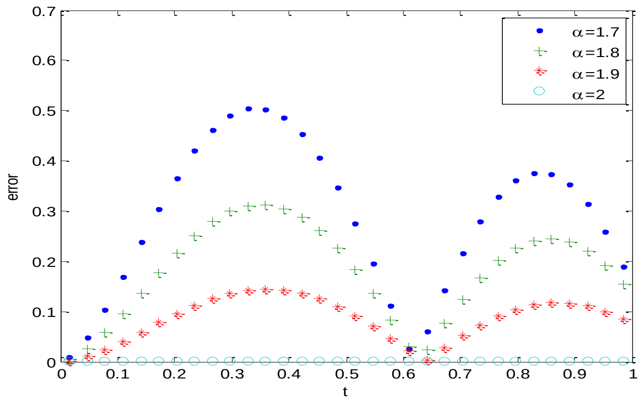

Figure 8. Error analysis for different values of α (Example 4)

5. Conclusion

In this study, we applied the Haar wavelet collocation method (HWCM) to solve linear and nonlinear fractional differential equations (FDEs). A general formulation for the Haar wavelet operational matrix of integration of fractional order was used to approximate the numerical solution of FDEs. The obtained numerical results agree with exact solutions, as α approaches an integer value, the method gives solutions for integer-order differential equations. The solutions obtained using the present technique show that this methodology can solve the problems more efficiently.

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