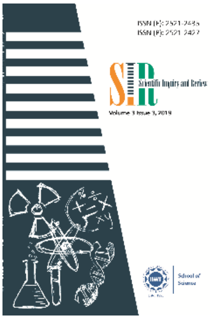

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Solution of Parabolic Partial Differential Equations via Non-Polynomial Cubic Spline Technique

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Abstract

The discovery of parabolic partial differential equation (PDE) has made a profound impact on the scientific, engineering and technological community. A vast amount of research has been conducted to find the solution of parabolic PDEs. In this research, we introduced a novel technique to find the numerical solution of the fourth order PDEs. The novel technique is based upon the polynomial cubic spline method (PCSM) used along with Adomian decomposition method (ADM). The constraint of the alternative variables was decomposed by ADM to achieve successive approximation. Additionally, a numerical test problem of parabolic PDEs was solved by the proposed technique to check its viability.

Keywords: Adomian decomposition method (ADM), fourth order parabolic PDEs, NPCS technique

Introduction

The field of differential equations first appeared in a calculus study conducted by the German mathematician Leibniz (1646-1716) and Newton (1642-1727). Newton found elementary laws of mechanics and calculus, which he shared privately with his friends. He was very conscious about disapproval and did not publish his results until 1687. He also worked on differential equations, but it was his laws of calculus and mechanics that offered a foundation for differential equations and their applications in the eighteenth century. The first order differential equation was classified by Newton as $\frac{dy}{dx} = f(y)$ and $\frac{dy}{dx} = f(x, y)$. Then, the solution was developed by him using the infinite series where $f(x,y)$ is a polynomial in x and y. Near the beginning of 1690, Newton's research on mathematics finished. His solution of irregular challenge problems and their results was published much earlier.

In this research work, we only focused on parabolic partial differential equations to introduce a novel technique to find the numerical solution of the fourth order PDEs. Parabolic partial differential equation has a great impact on our scientific, engineering, and technological community. Heat conduction, reverse heat problem, thermal conductivity, convection diffusion equations, and Fokker-plank equations are all types of parabolic partial differential equations. After looking into the findings of prior research, we understand that partial differential equations are essential for the development of physical models based on vibration of strings, electric fields, gravitational fields, and heat problems. In the field of differential geometry, partial differential equations particularly parabolic partial differential equations play a vital role in the Riemann's application of a potential theoretic argument, the Dirachlet principle and its uses, in developing the general theory of analytic functions of a complex variable, and the related theory of Riemann surfaces. PDE's also act as a bridge between central mathematical issues and practical applications that take place in the field of probabilistic models, such as in the case of the so-called stochastic processes. PDEs rose in importance through the study of Brownian motion. It was expanded by Ito, Levy, Kolmogorov, and several other mathematicians who made it into a general theory of stochastic differential equations. Such type of problems were solved by different mathematicians, for example, [1] Cagler, S.H. used the non-polynomial cubic spline method to solve the time dependent parabolic heat problem. [2] Omotayo andOgunian used a non-polynomial cubic spline method to solve the linear fourth order parabolic problem. [3] Evans use of AGE method for the fourth order parabolic equation [4, 5, 6].

In present day and age, it has become easier to compute the value of a function, draw a graph, and finding the error. One of the interpolation functions is spline interpolations. In their most general form, splines can be considered a mathematical model that depicts a continuous representation of a curve or a surface with a discrete set of points in a given space. Spline fitting is an extremely popular form of piecewise approximation that uses various polynomials of degree n, or more general functions, during an interval in which they are fitted to a function at specified points known as control points, nodes or simply knots. The polynomial used can be changed, but the derivatives of the polynomials are required to match up to the degree $n-1$ at each side of the knots or meet related interpolatory conditions. Boundary conditions are also imposed on the end points of the intervals.

The stander equation of linear parabolic partial differential equation is

$$u_{tt}(x,t) + u_{xxxx}(x,t) = h(x,t)\, 0 \leq x \leq 1, t \geq 0 \tag{1}$$

with initial conditions (ICs) being

$$u(x,0) = f(x)\,, \qquad u_t(x,0) = g(x) \tag{2}$$

and boundary conditions (B.C.) being

$$u(0,t) = \alpha(t), \qquad u(1,t) = \beta(t) \tag{3}$$

In the equation (1) h(x, t) sourcing function, x and t are the flow of heat and time variables, respectively. Where $f(x), \alpha(t), \beta(t)$, and g(x) are the continuous function of variables x and t as denoted in (2) and (3). Many PDEs [7, 8, 9] have been solved by polynomial and non-polynomial cubic spline technique. Papamichael and Worsley [10] found the solution of boundary value problems with the help of cubic spline method of order four. In another study, a numerical solution presented by [11] was employed with the help of non-polynomial cubic spline technique for solving the fourth order linear boundary value problem. Various numerical methods have been used to solve the parabolic partial differential equations. In 2008, some authors evaluated the convergence of cubic spline approach to find a solution for the boundary value problem. They formed the numerical solution of sixth and twelfth order ODES boundary value problems by means of non-polynomial cubic spline methods [11, 12]. In a like manner, the major objective of this research article is to find out the numerical solution of fourth order parabolic partial differential equation using a non-polynomial cubic spline.

1. Mathematical preliminaries

The equation (1) was converted into a system of PDEs by using method given below:

Let,

$$u_t = w \tag{4}$$

$$\Longrightarrow\ u_{tt} = w_t$$

Let,

$$u_{xx} = v$$

$$u_{xxxx} = v_{xx} \tag{5}$$

Equation (1) can be noted as

$$v_{xx} = -w_t + h(x,t) \tag{6}$$

(5) and (6) along with the I.C.

$$u(x,0) = f(x)\,, \qquad v(x,0) = g(x)$$

a system of PDEs with B.C. in equation (3) form which is to be resolved by non-polynomial cubic spline method.

2.1. Formulating of NPCS Technique

For the construction of NPCS technique S for equation (1) and under the boundary values in equation (3), the interval [0, 1] was discretized by equally spaced knots:

$x_j = x_0 + jh, \quad j = 0,1,\dots,n$, where, $x_0 = 0, x_n = 1$ and $h = \frac{1}{n}$.

For each segment [x_j, x_{j+1}], we consider a non-polynomial spline $S_j(x)$, j=0, 1…. n, which is written as follows:

$$S_j(x) = a_j + b_j(x - x_j) + c_j sink(x - x_j) + d_j cosk(x - x_j), j = 0, 1, \dots, n-1 \qquad (7)$$

Here, a_j, b_j, c_j, and d_j represents arbitrary constants, whereas free parameter is represented by k. We assume that u_j is an estimation of $u(x_j)$ that was given by the segment $S_j(x)$ of the NPCS traversing the two points (x_j, u_j) and (x_{j+1}, u_{j+1}). By $S_j(x)$ at both points, i.e., x_j and $x_{j+1}$are satisfied by the interpolation condition such as given in equation (3), while the boundary condition and the continuity condition of the first derivative are satisfied at grid points (x_j, u_j).

Let, $S_j(x_j) = M_j, S_j(x_{j+1})=M_{j+1}, S_j''(x_j) = L_j,\ S_j''(x_{j+1}) = L_{j+1}$ (8)

Now using the first interpolation condition, namely,

$S_j(x_j) = U_j$, and setting $x = x_j$ in equation (7), we have,

$S_j(x_j) = a_j + b_j(x_j - x_j) + c_j sink(x_j - x_j) + d_j cosk(x_j - x_j),\ j = 0, 1, \dots, n-1$

$S_j(x_j) = a_j + d_j$

where from the equation (8) it is determined that

$$M_j = a_j + d_j \tag{9}$$

Now, equation (7) becomes

$S_j(x_{j+1}) = a_j + b_j(x_{j+1} - x_j) + c_j\ sink(x_{j+1} - x_j) + d_j\ cosk(x_{j+1} - x_j)$

Again using equation (8) and replacing $x_{j+1} - x_j = h$ (the length of the interval) we have

$$M_{j+1} = a_j + b_j h + c_j sinkh + d_j coskh \tag{10}$$

Now, for the condition of continuity of the slop of the curve, we differentiate the non-polynomial spline $S_j(x)$ defined in equation (7), with respect to x,

$$S_j'(x) = b_j + kc_j Cosk(x - x_j) - kd_j Sink(x - x_j) \tag{11}$$

Now, for the slope of the curve at point x_j

$$S_j'(x_j) = b_j + kc_j \tag{12}$$

Again, for the slope of the curve at point x_{j+1}

$S_j'(x_{j+1}) = b_j + kc_j\ coskh - kd_j\ sinkh$

$$b_{j+1} + kc_{j+1} = b_j + kc_j\ coskh - kd_j\ sinkh, \tag{13}$$

Now, the curvature of the non-polynomial cubic spline function, is given by

$$S_j''(x) = -k^2 c_i\ sink(x - x_j) - k^2 d_i\ cosk(x - x_j) \tag{14}$$

At (x_j, x_{j+1}), we put $x = x_j$ and $x = x_{j+1}$ in equation (14).

$$L_j = -k^2 d_j \tag{15}$$

$$\Rightarrow d_j = -\frac{L_j}{k^2} \tag{16}$$

$$L_{j+1} = -k^2 c_j \, sinkh - k^2 d_j \, coskh \tag{17}$$

From equation (15), we have the following equation:

$$d_{j+1} = c_j \, sinkh + d_j \, coskh \tag{18}$$

Now, we will make suitable substitutions in equations (9-18) to determine the remaining unknown coefficients a_j, b_j, and c_j, respectively.

If we substitute the value of d_j from equation (16) to equation (9), we have

$$M_j = a_j - \frac{L_j}{k^2}, \Rightarrow a_j = M_j + \frac{L_j}{k^2} \tag{19}$$

Now, equation (17) can be solved to find c_j.

$$c_j = \frac{1}{k^2 Sin\theta}(L_j cos\theta - L_{j+1}) \tag{20}$$

Finally, for b_j, if we subsitute the values of a_j, b_j, and c_j in equation (10), we have

$$b_j = \frac{1}{h}(M_{j+1} - M_j) - \frac{1}{hk^2}(L_j - L_{j+1}) \tag{21}$$

Hence, all the unknowns such as a_j, b_j, c_j, and d_j are find out and are shown in equations (19-21) and (16), respectively.

Now, we will use the continuity condition of the first derivative at grid point (u_j, x_j) to consistency relationship, also known as recurrence relationship.

For this, we take the continuity of the spline function $S_j(x)$ at point x_j as,

$$S'_{j-1}(x_j) = S'_j(x_j) \tag{22}$$

Also, we take the non-polynomial cubic spline function $S_j(x)$ in the interval $[x_j, \ x_{j+1}]$ from equation (7),

$$S_j(x) = a_j + b_j(x - x_j) + c_j sink(x - x_j) + d_j cosk(x - x_j), j = 0, 1, \dots, n-1$$

similarly, we can write the spline function $S_{j-1}(x)$ in the interval $[x_{j-1},\ x_j]$ as

$$S_{j-1}(x) = a_{j-1} + b_{j-1}(x - x_{j-1}) + c_{j-1} sink(x - x_{j-1}) + d_{j-1} cosk(x - x_{j-1}) \tag{23}$$

From equation (23) we have the following equation:

$$S'_{j-1}(x) = b_{j-1} + kc_{j-1} cosk(x - x_{j-1}) - kd_{j-1} sink(x - x_{j-1}) \tag{24}$$

Now from equation (24), we have the following equation:

$$S'_{j-1}(x_j) = b_{j-1} + kc_{j-1} cos\theta - kd_{j-1} sin\theta \tag{25}$$

Now, from equation (22), we have the following equation:

$$b_{j-1} + kc_{j-1} cos\theta - kd_{j-1} sin\theta = b_j + kc_j \tag{26}$$

Where, b_{j-1}, c_{j-1} and d_{j-1} are the unknown coefficients for spline function $S_{j-1}(x)$, whose values can be determined by similar pattern as for spline function $S_j(x)$.

The values of unknown coefficients b_{j-1}, c_{j-1}, and d_{j-1} are given as follows:

$$b_{j-1} = \frac{1}{h}(M_j - M_{j-1}) - \frac{1}{hk^2}(L_{j-1} - L_j),\ c_{j-1} = \frac{1}{k^2 Sin\theta}(L_{j-1} cos\theta - L_j),\quad d_{j-1} = -\frac{L_{j-1}}{k^2} \tag{27}$$

Replacing the values of unknown coefficients from equation (27), and the values of b_j and c_j from equation (20) and (21) in equation (26), we deduce/find the following equation:

$$\frac{1}{h^2}(M_{j-1} - 2M_j + M_{j+1}) = \alpha L_{j-1} + 2\beta L_j + \alpha L_{j+1} \tag{28}$$

Given that:

$\alpha = \left(\frac{1}{\theta Sin\theta} - \frac{1}{\theta^2}\right)$, and $\beta = \left(\frac{1}{\theta Sin\theta} - \frac{1}{\theta^2}\right)$

The equation (28), for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ fulfill the condition $1 - 2\alpha - 2\beta = 0$, proves that the established technique is fourth-order convergent.

Now, the equation (28) is a definitive draft of NPCS technique for the PDEs. It is ready and could be applied to the system of PDEs formulated at the common nodes(x_i, u_i) in equation (4) and (5)

2.2. Implementation of NPCS Technique for 4th Order Parabolic PDEs

Take $u_{xx} = u'' = L_j$ and by means of the central finite difference approximations of $O(h^2)$ for the first order time derivatives u_t and w_t, we obtained the following equation:

$$u_t = u_j' \cong \frac{u_j - u_{j-1}}{k} \text{ and } w_t = w_j' \cong \frac{w_j - w_{j-1}}{k} \tag{30}$$

Substitute the values of u_t and v_t from equation (30) in equation (4) and (5), we obtained

$$\frac{u_j - u_{j-1}}{k} = v_j \text{ and} \tag{31}$$

$$L_j = \frac{w_j - w_{j-1}}{-k} + h(x,t) \tag{32}$$

Equations (31) and (32) could be written as follows:

$$u_j - kw_j = 0, \tag{33}$$

$$L_j = -\frac{1}{k}\left(w_j - w_{j-1}\right) + h(x,t) \tag{34}$$

Approximating $u_{j-1} = f_j$ and $v_{j-1} = g_j$, then equations (33) and (34) were are as follows:

$$u_j - kw_j = 0, \tag{35}$$

$$L_j = -\frac{1}{k}\left(w_j - g_j\right) + h(x,t) \tag{36}$$

Now, from equation (36) we obtained

$$L_{j+1} = -\frac{1}{k}\left(w_{j+1} - g_{j+1}\right) + h(x,t) \text{ and} \tag{37}$$

$$L_{j-1} = -\frac{1}{k}\left(w_{j-1} - g_{j-1}\right) + h(x,t) \tag{38}$$

Using equations (36-38) in equation (29), we obtained

$$\frac{1}{h^2}\left(M_{j-1}-2M_j+M_{j+1}\right)=\alpha(\frac{1}{k}\left(w_{j-1}-g_{j-1}\right)+h(x,t))+2\beta(\frac{1}{k}\left(w_j-g_j\right)+h(x,t))+\alpha(\frac{1}{k}\left(w_{j+1}-g_{j+1}\right)+h(x,t)) \quad (39)$$

$$\Longrightarrow M_{j+1}\left(\alpha-\frac{1}{h^2}\right)+2M_j\left(\beta+\frac{1}{h^2}\right)+M_{j-1}\left(\alpha-\frac{1}{h^2}\right)+\alpha\left(\frac{w_{j-1}-g_{j-1}}{k}\right)+2\beta\left(\frac{w_j-g_j}{k}\right)+\alpha\left(\frac{v_{j+1}-g_{j+1}}{k}\right)-2(\alpha+\beta)h(x,t)=0 \quad (40)$$

The equations (35) and (40) form a comprehensive system of algebraic equations. These are associated with the BCs given in equations (3) and (6). Simple technique equations could be used to solve it.

3. Results & Discussion

3.1. Non-Homogeneous test problems

Test problem 1

Consider

$$u_{tt}+u_{xxxx}=2e^{x+t}$$

with initial conditions

$u(x,0)=e^x\ and\ u_t(x,0)=e^x$

with exact solution

$$u(x,t)=e^{x+t}$$

Table 1. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

X	**Exact**	**PCSM**	**Absolut error for PCSM**	**NPCSM**	**Absolute error for NPCSM**
0.2	1.3504	1.376811352	2.69525E-02	1.34986	5.4000E-04
0.4	1.65223	1.693969607	4.52483E-02	1.64872	3.5100E-03
0.6	2.01596	2.062112298	4.83596E-02	2.01375	2.2100E-03
0.8	2.45564	2.491047203	3.14441E-02	2.4596	3.9600E-03

Table 2. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

X	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	1.23368	1.222641923	6.9618E-03	1.23373	5.0000E-05
0.4	1.50682	1.493456733	7.6682E-03	1.5069	8.0000E-05
0.6	1.84043	1.823600371	1.2637E-02	1.84054	1.1000E-04
0.8	2.24791	2.228154275	1.8360E-02	2.24801	1.0000E-04

Table 3. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

X	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	1.22262	1.222831125	1.7153E-05	1.22263	1.000E-05
0.4	1.4933172	1.493285989	1.3965E-04	1.49332	2.800E-06
0.6	1.823941831	1.823885797	3.4146E-04	1.82394	1.835E-06
0.8	2.227767	2.228337192	3.8662E-04	2.22777	3.000E-06

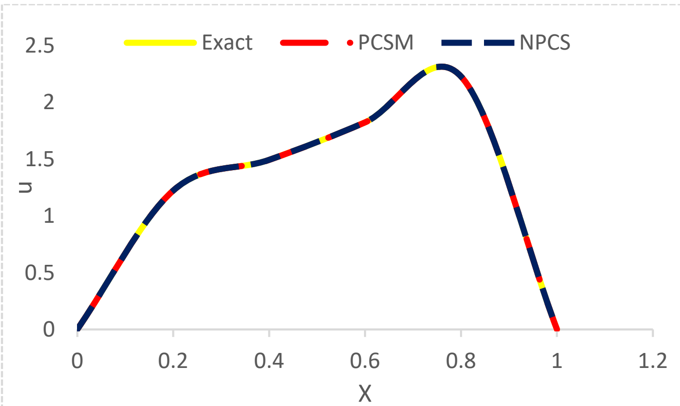

Figure 1. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1The superiority of purposed method is clearly shown in this graph

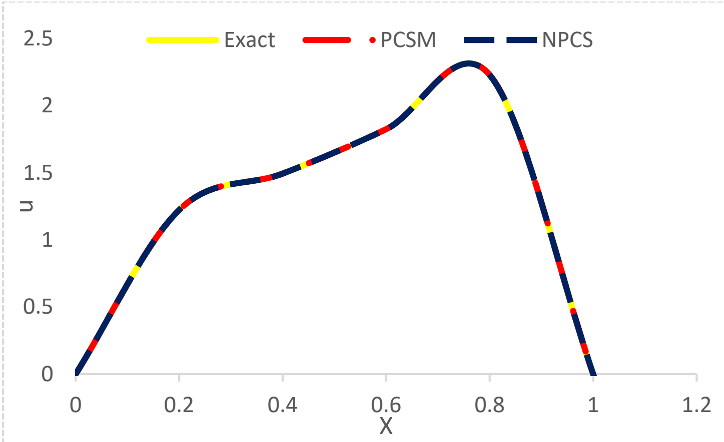

Figure 2. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01. The superiority of purposed method is clearly shown in this graph

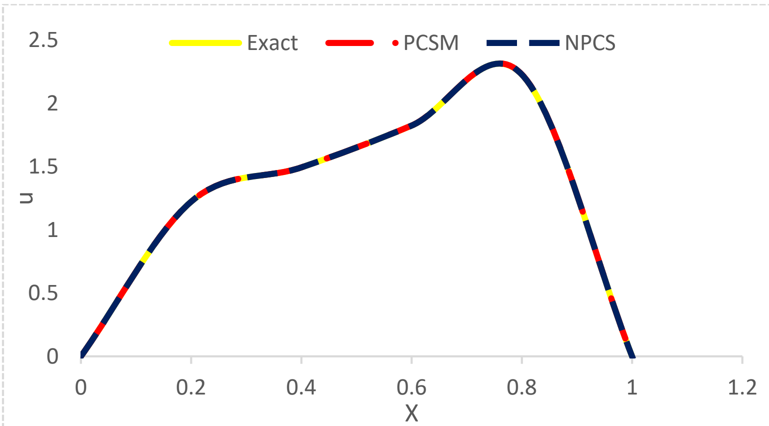

Figure 3. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001. The superiority of purposed method is clearly shown in this graph

Test problem 2

Consider

$$u_{tt} + u_{xxxx} = 2e^{x-t}$$

with initial conditions

$u(x,0) = e^{x}\ and\ u_t(x,0) = -e^{x}$

with exact solution

$$u(x,t) = e^{x-t}$$

Table 4. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

X	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	1.105170918	1.167323488	3.9230E-03	1.104880041	1.1000E-04
0.4	1.349858808	1.451263874	1.9245E-03	1.351839629	9.0000E-05
0.6	1.648721271	1.756777068	1.0805E-01	1.649431071	7.0980E-04
0.8	2.013752707	2.08764511	7.3892E-02	2.009169441	4.5832E-03

Table 5. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

X	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	1.105170918	1.167323488	3.9230E-03	1.104880041	1.1000E-04
0.4	1.349858808	1.451263874	1.9245E-03	1.351839629	9.0000E-05
0.6	1.648721271	1.756777068	1.0805E-01	1.649431071	7.0980E-04
0.8	2.013752707	2.08764511	7.3892E-02	2.009169441	4.5832E-03

Table 6. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

X	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	1.20925	1.216072764	4.1092E-03	1.2093	5.0000E-05
0.4	1.47698	1.484496211	5.8374 E-03	1.47706	8.0000E-05
0.6	1.80399	1.816374581	3.9230 E-03	1.8041	1.1000E-04
0.8	2.2034	2.221391945	1.9245 E-03	2.20349	9.0000E-05

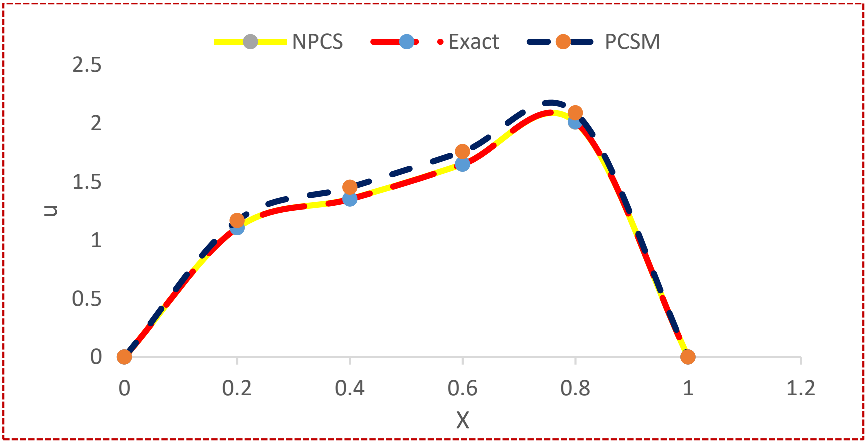

Figure 4. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0. 1. The superiority of purposed method is clearly shown in this graph

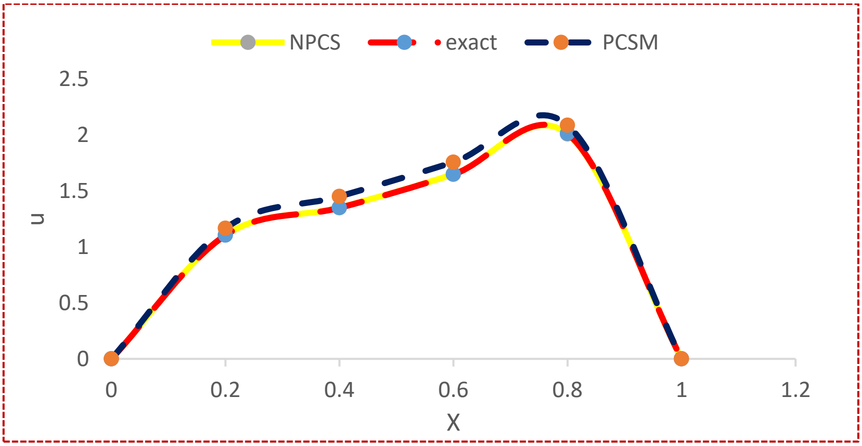

Figure 5. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01. The superiority of purposed method is clearly shown in this graph

In this work, the validity of the polynomial and non-polynomial cubic spline method was checked through its application to a variety of test problems. The results were compared with some already existed results in literature. In above given problem 1, we applied polynomial as well as the non-polynomial cubic spline method. The results were also compared with the exact solution and are shown in Table (1-3) and figure (1-3) at the spatial step h=$\frac{1}{5}$ and temporal step size $k = 0.1, 0.01, 0.001$. The superiority of the non-polynomial cubic spline method can be seen clearly in Table (1-3). Polynomial cubic spline method obtained minimum absolute error of 1.9245×10^{-3} at x=0.4 and k = 0.1 at 1.924555×10^{-3} as shown in Table 1 and Table 3, respectively. On the other hand the minimum absolute error obtained by the non-polynomial cubic spline method reaches to 1.9×10^{-5} as can be observed in the Table (1-3). A similar kind of experiment is performed in problem 2, at h= $\frac{1}{5}$ and k=0.1, 0.01, 0.001. The results are shown in the Table (4-6) and Figure (4-6). Again, here the upper hand of the non-polynomial cubic spline method can be observed candidly. The minimum absolute error obtained by PCSM is 1.8×10^{-1} at h=$\frac{1}{5}$, k=0.1 as shown in Table 4, while the minimum error by NPCS reaches to 0 at x=0.4, k=0.001 as shown in Table 6).

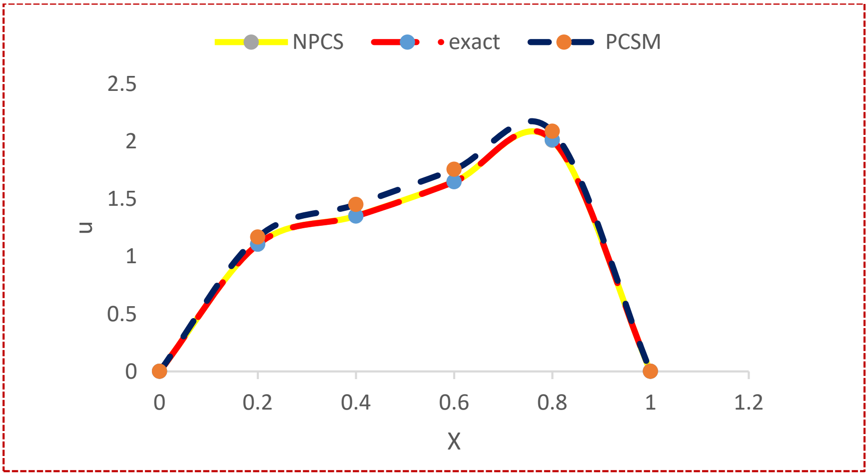

Figure 6. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

Overall, NPCSM is found to be the better as compare to PCSM and some already existing methods and results. NPCSM provides better results for through smaller time steps. A slight betterment was observed with the decrease in spatial step size h.

4. Conclusion

The main focus in this research article was to develop an interpolation technique for the solution of the fourth order parabolic partial differential equations (PDEs). Numerous techniques are available in literature and are used to solve the ordinary differential equations and partial differential equations. In this study, the proposed method is non-polynomial cubic spline method (NPCSM). It was used to solve the fourth order parabolic partial differential equations.

The numerically obtained results were compared with polynomial cubic spline method (PCSM) and also with some previously existing methods in the literature [4, 6]. The numerical results were verified at different time steps and spatial intervals through a comparison with the exact solution.

The validity of the method was checked through test problems. The superiority of the constructed technique can be seen in problems (1-4), where the numerical results obtained by the non-polynomial cubic spline method are compared with the numerical results obtained by the polynomial cubic spline method.

In test problems (5-8), the results were compared with [4, 5], show better results. Additionally, it was observed that the method is convergent and the accuracy in the results increases with the decrease in the length of temporal and spatial intervals. Furthermore, the technique can be used to solve the various other higher order linear differential equations.

The proposed method depends on the defined non-polynomial cubic spline methods. However, future researchers may obtain other non-polynomial cubic spline methods so the results may be improved further.

Conflict of Interest

The authors declare no conflict of interest.

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