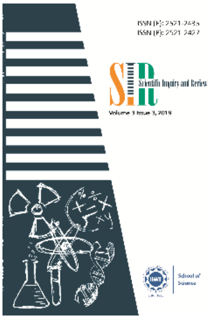

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Author(s): Bilal Ahmad[1], Anjum Perviaz[2], Ozair Ahmad[1], Fazal Dayan[1]

Affiliation: [1]Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
[2]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

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Numerical Solution of Fourth Order Homogeneous Parabolic Partial Differential Equations (PDEs) using Non-Polynomial Cubic Spline Method (NPCSM)

Bilal Ahmad[1*], Anjum Perviaz[2], Ozair Ahmad[1], Fazal Dayan[1]

[1]Department of Mathematics and Statistics,
The University of Lahore, Lahore, Pakistan
[2]Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan

[*]bilal.math@outlook.com

Abstract

Non-polynomial cubic spline functions are already being used in the field of engineering, computer sciences, and natural sciences to solve ordinary differential equations (ODEs) and partial differential equations (PDEs). However, many of the above-mentioned problems do not have an exact, stable, or convergent exact solution. There are different approximations and methods that can be applied to solve these problems. This study implemented the purposed method on homogeneous parabolic PDEs having different dimensions. The results obtained were compared with the exact solution and results of other existing methods in tabular and graphical form. Mathematica was used to find the mathematical and graphical results.

Keywords: Adomian decomposition method (ADM), non-polynomial cubic spline method (NPCSM), continuous approximation, finite difference approximations, fourth order homogeneous parabolic partial differential equations (PDEs)

Introduction

Splines are used to model a curve using a group of points that can be mapped by mathematical technique. The impact of weight is maximum at the point of contact between points. As the points move farther apart, the weight diminishes along the spine. Hence, the impact, dimension, and physical relation of every weight is adjusted with the help of spline. In aircraft industries, splines were not only used for modelling designs but were also used to determine flight trajectories. Depending upon the different features and scenarios, different kinds of spline were used in the construction of a mathematical design [1]. There are many types of spline; however, this research focused on the non-polynomial cubic spline method (NPCSM). In the past, many complicated functions were used for estimation of value of non-polynomial cubic spline such as logarithmic trigonometric, statistical density functions. Presently, it is easier to compute the value of a function, draw a graph, and determine an error due to the use of mathematical methods [2]. One of the interpolation functions is spline interpolations. The rapid development of spline functions is primarily due to its vast usage in approximating the solutions of a variety of problems, arising in engineering and applied mathematics. The classes of spline functions possess many agreeable structural properties as well as an excellent approximation ability. Splines and their applications have been effectively used in data fitting, optimal control problems, function approximation, integrodifferential equation, Computer-Aided Geometric Design (CAGD), and wavelets. Many programs, based on spline functions, are being used in many computer applications [3]. Originally, splines were thin wooden or metal strips that were used in shipbuilding and aircraft industries to create smooth curves. Naval architects used thin splints pulled into place by weights (called knots or ducks). The wooden strips (splines) were flexible enough to bend accordingly when a weight was placed or removed. Boat-builders used to add more weights on a certain region of the spline to bend them where needed. Boats and ships have been built using this method for centuries. When conducting an experiment on the model of an airplane, using mathematical model for calculation can shorten the duration needed to measure each section of the plane. However, unlike physical models, mathematical methods would require data exchange to illustrate the shape of the curve [4].

This study focused on examining parabolic partial differential equations (PDEs). Parabolic PDE significantly impacts the field of science, engineering, and technology. Heat conduction, reverse heat problem, thermal conductivity, convection-diffusion equations, and Fokker-plank equations are all a type of parabolic PDE. Due to their significant use, PDE became important in the development of physical models, such as the vibration of strings, electric fields, gravitational fields, and heat problems. In the field of differential geometry, parabolic PDE play a vital role. These problems were solved by different mathematicians, such as Ahmad et al, 2021 [1], Taiwo and Ogunlaran, 2011 [2], and Katz and Fridman, 2020 [5]. Many other studies have also solved the constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs. [6, 7, 8] sampled the data for 1-D parabolic PDEs with non-local outputs. Pervaiz and Ahmad also attempted to solve fourth-order parabolic two-point boundary value problems by using cubic spline method [9]. Few researchers, such as [10-14], also worked on the spline technique to solve different types of differential equations.

Mathematically, a spline function comprises polynomial pieces on subintervals that are joined together with certain continuity conditions. A spline function of degree k, having knots $x_0, x_1, x_2, \dots, x_n$, is a function of S such that:

- S is a polynomial of degree $\leq k$ on each interval $[x_{i-1}, x_i]$;
- S has a continuous $(k-1)^{st}$ derivative on $[x_0, x_n]$.

The major objective of this study is to determine the numerical solution of the fourth order homogenous parabolic partial differential equation.

2. Methods and Formulation

A cubic spline $S_{3,n}(x)$ is a C^2 class piecewise cubic polynomial. Thus, for cubic spline:

- $S_{3,n}(x)$ is a piecewise cubic between successive knots x_i.

$$S_{3,n}(x) = \begin{cases} p_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, & x \in [x_0, x_1], \\ p_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3, & x \in [x_1, x_2], \\ \quad\quad . \\ \quad\quad . \\ p_n(x) = a_n + b_n x + c_n x^2 + d_n x^3, & x \in [x_{n-1}, x_n]. \end{cases}$$

- $S_{3,n}(x)$ is a class C^2, such that, $S_{3,n}(x)$ is continuous and has continuous first and second derivatives all over in the interval $[a, b]$, in particular, at the knots.

Also, for cubic spline interpolation:

- $S_{3,n}(x)$ interpolates the data, that is,

$$S_{3,n}(x_i) = f_i, \quad i = 0, 1, \dots, n.$$

The cubic polynomial is continuous for each subinterval (x_{i-1}, x_i) and has continuous n - order derivatives. For this reason, in the entire interval $[a, b]$, cubic spline or one of its derivatives can be discontinuous only at the place where these cubic polynomial pieces unite. Thus, we have to test the continuity at every knot x_i, as well as the continuity of first and second derivatives at each x_i in the interval $[a, b]$.

The continuity conditions for cubic spline at each knot x_i can be written as

$p_i'(x_i) = p_{i+1}'(x_i)$ (by the condition of continuity of first order derivative) and

$p_i''(x_i) = p_{i+1}''(x_i)$ (by the condition of continuity of first second derivative).

Where $i = 0,1,2, \dots, n - 1$.

Subsequently, we use the interpolatory conditions. These conditions are the function evaluation condition which will be the same for quadratic, cubic, or any spline of any order. That is,

$p_i(x_{i-1}) = f_{i-1}$, and $p_i(x_i) = f_i$, where $i = 0,1,2, \dots, n$.

As every cubic piece has four unknown coefficients; therefore, $S_{3,n}(x)$ has $4n$ unknown coefficients. We have to find these unknown coefficients in the cubic spline. Due to the continuity of the first and second derivatives, we get $3(n - 1)$ linear constraints. Moreover, interpolation imposes extra $n + 1$ linear constraints. For this reason, we have total $4n - 2n$ linear constraints. We require two additional constraints since we have $4n - 2n$ linear constraints for $4n$ unknown coefficients.

There are different methods to find two more constraints. For instance:

- When we apply the conditions, the natural cubic spline gives

$$p_1''(x_0) = 0, \qquad p_n''(x_n) = 0.$$

- Instead of using natural cubic spline conditions, we can use the correct second derivative values

$$p_1''(x_0) = f''(x_0), \qquad p_n''(x_n) = f''(x_n).$$

If values of the second derivatives are not obtainable, then they can be changed by precise approximations.

- A clear-cut and precise spline can be obtained by using the knot conditions. The major scheme of the knot conditions is that the cubic polynomials remain the same when piecewise polynomials cross both the first and last interior nodes, x_1 and x_{n-1}. These conditions can be expressed mathematically as

$p_1'''(x_0) = p_2'''(x_1),\ p_{n-1}'''(x_{n-1}) = p_n'''(x_{n-1}).$

The former two piecewise cubic polynomials $p_1(x)$, and $p_2(x)$ of cubic spline agree with the first and second derivatives at the knot x_1 if $p_1(x)$ and $p_2(x)$ also satisfy the conditions.

- In the whole cubic spline interpolation, the gradient conditions are given as

$$p_1'(x_0) = f'(x_0), \qquad p_n'(x_n) = f'(x_n)$$

are imposed. These first derivative estimations of the data may not be quickly available but they can be replaced by accurate approximations.

2.1. Construction of NPCS Technique

For the construction of NPCS technique, S is taken for equation (1) under the BC in equation (3), while the interval [0, 1] was divided in equal parts as

$x_j = x_0 + jh, \quad j = 0,1,\dots,n,$ where, $x_0 = 0, x_n = 1$ and $h = \frac{1}{n}$.

For each segment [x_j, x_{j+1}], we considered a non-polynomial spline $S_j(x)$, j=0, 1…. n, which is written as

$S_j(x) = a_j + b_j(x - x_j) + c_j sink(x - x_j) + d_j cosk(x - x_j), j = 0, 1, \dots, n-1,$ (7)

where a_j, b_j, c_j, and d_j are arbitrary constants and k is a free parameter.

Let $S_j(x_j) = M_j$, $S_j(x_{j+1}) = M_{j+1}$, $S_j''(x_j) = L_j$, and $S_j''(x_{j+1}) = L_{j+1}$. (8)

Using the first interpolation condition, that is,

$S_j(x_j) = U_j$, and setting $x = x_j$ in equation (7), we had

$$S_j(x_j) = a_j + b_j(x_j - x_j) + c_j sink(x_j - x_j) + d_j cosk(x_j - x_j),$$
$$j = 0, 1, \ldots, n-1,$$

$S_j(x_j) = a_j + d_j,$

where from the equation (8),

$M_j = a_j + d_j.$ (9)

Equation (7) becomes

$$S_j(x_{j+1}) = a_j + b_j(x_{j+1} - x_j) + c_j\ sink(x_{j+1} - x_j) + d_j\ cosk(x_{j+1} - x_j),$$

Again, after using equation (8) and replacing $x_{j+1} - x_j = h$, the length of the interval, we determined

$M_{j+1} = a_j + b_j h + c_j sinkh + d_j coskh.$ (10)

To find the condition of continuity of the slop of the curve, we differentiated the non-polynomial spline $S_j(x)$ defined in equation (7) with respect to x as

$S_j'(x) = b_j + kc_j Cosk(x - x_j) - kd_j Sink(x - x_j).$ (11)

To find the slope of the curve at point x_j,

$S_j'(x_j) = b_j + kc_j.$ (12)

Again, to find the slope of the curve at point x_{j+1},

$S_j'(x_{j+1}) = b_j + kc_j\ coskh - kd_j\ sinkh,$

but from equation (12),

$b_{j+1} + kc_{j+1} = b_j + kc_j\ coskh - kd_j\ sinkh.$ (13)

Hence, the curvature of the non-polynomial cubic spline function was determined as

$S_j''(x) = -k^2 c_i \, sink(x - x_j) - k^2 d_i \, cosk(x - x_j).$ (14)

At (x_j, x_{j+1}), we put $x = x_j$ and $x = x_{j+1}$ in equation (14).

$L_j = -k^2 d_j,$ (15)

$d_j = -\frac{L_j}{k^2}$ (16)

$L_{j+1} = -k^2 c_j \, sinkh - k^2 d_j \, coskh.$ (17)

From equation (15), we had

$d_{j+1} = c_j \, sinkh + d_j \, coskh.$ (18)

Next, we made suitable substitutions in equations (9-18) to determine the remaining unknown coefficients a_j, b_j and c_j, respectively.

First, we substituted the value of d_j from equation (16) to equation (9) to get

$M_j = a_j - \frac{L_j}{k^2} \Longrightarrow a_j = M_j + \frac{L_j}{k^2}$ (19)

Now, equation (17) can be solved to find c_j.

$c_j = \frac{1}{k^2 Sin\theta}(L_j cos\theta - L_{j+1}).$ (20)

Finally, for b_j, we substituted the values of a_j, b_j, and c_j in equation (10) to get

$b_j = \frac{1}{h}(M_{j+1} - M_j) - \frac{1}{hk^2}(L_j - L_{j+1}).$ (21)

Hence, all the unknowns a_j, b_j, c_j, and d_j were determined and are shown in equations (19-21) and (16), respectively.

Subsequently, we used the continuity condition of the first derivative at grid point (u_j, x_j) to check consistency relationship, also known as recurrence relationship.

For this, we took the continuity of the spline function $S_j(x)$ at point x_j as

$S_{j-1}'(x_j) = S_j'(x_j)$ (22)

Next, we took the non-polynomial cubic spline function $S_j(x)$ in the interval $[x_j,\ x_{j+1}]$ from equation (7),

$$S_j(x) = a_j + b_j(x - x_j) + c_j sink(x - x_j) + d_j cosk(x - x_j), j = 0, 1, \ldots, n - 1.$$

Similarly, we wrote the spline function $S_{j-1}(x)$ in the interval $[x_{j-1},\ x_j]$ as,

$$S_{j-1}(x) = a_{j-1} + b_{j-1}(x - x_{j-1}) + c_{j-1} sink(x - x_{j-1}) + d_{j-1} cosk(x - x_{j-1}). \quad (23)$$

From equation (23) we had

$$S'_{j-1}(x) = b_{j-1} + kc_{j-1} cosk(x - x_{j-1}) - kd_{j-1} sink(x - x_{j-1}). \quad (24)$$

Afterwards from equation (24), we had

$$S'_{j-1}(x_j) = b_{j-1} + kc_{j-1} cos\theta - kd_{j-1} sin\theta. \quad (25)$$

From equation (22), we had

$$b_{j-1} + kc_{j-1} cos\theta - kd_{j-1} sin\theta = b_j + kc_j, \quad (26)$$

Where, b_{j-1}, c_{j-1} and d_{j-1} are the unknown coefficients for spline function $S_{j-1}(x)$, whose values can be determined by a similar pattern as for spline function $S_j(x)$.

The values of unknown coefficients b_{j-1}, c_{j-1}, and d_{j-1} are given as follows:

$$b_{j-1} = \frac{1}{h}(M_j - M_{j-1}) - \frac{1}{hk^2}(L_{j-1} - L_j),\ c_{j-1} = \frac{1}{k^2 Sin\theta}(L_{j-1} cos\theta - L_j),\quad d_{j-1} = -\frac{L_{j-1}}{k^2}. \quad (27)$$

We replaced the values of unknown coefficients from equation (27) and values of b_j and c_j from equation (20) and (21) in equation (26),

$$\frac{1}{h^2}(M_{j-1} - 2M_j + M_{j+1}) = \alpha L_{j-1} + 2\beta L_j + \alpha L_{j+1} \quad (28)$$

Where,

$$\alpha = \left(\frac{1}{\theta Sin\theta} - \frac{1}{\theta^2}\right), \text{ and } \beta = \left(\frac{1}{\theta Sin\theta} - \frac{1}{\theta^2}\right).$$

Hence, in equation (28), $\alpha. = \frac{1}{12}$ and $\beta. = \frac{5}{12}$ satisfying the condition $1. -2\alpha. -2\beta = 0$, implies that the developed scheme is fourth-order convergent.

2.2. Application of NPCS Technique for 4th Order Parabolic PDEs

We took $u_{xx} = u'' = L_j$ and used the central finite difference approximations of $O(h^2)$ for the first order time derivatives u_t and w_t, we had

$$u_t = u_j' \cong \frac{u_j - u_{j-1}}{k} \text{and } w_t = w_j' \cong \frac{w_j - w_{j-1}}{k}. \quad (30)$$

We substituted the values of u_t and v_t from equation (30) in equation (4) and (5) to obtain

$$\frac{u_j - u_{j-1}}{k} = v_j \quad (31)$$

$$L_j = \frac{w_j - w_{j-1}}{-k} + h(x,t) \quad (32)$$

Equations (31) and (32) could be written as

$$u_j - kw_j = 0 \quad (33)$$

$$L_j = -\frac{1}{k}\left(w_j - w_{j-1}\right) + h(x,t) \quad (34)$$

Approximating $u_{j-1} = f_j$ and $v_{j-1} = g_j$, then equations (33) and (34) were as under

$$u_j - kw_j = 0 \quad (35)$$

$$L_j = -\frac{1}{k}\left(w_j - g_j\right) + h(x,t) \quad (36)$$

From equation (36) we obtained

$$L_{j+1} = -\frac{1}{k}\left(w_{j+1} - g_{j+1}\right) + h(x,t) \quad (37)$$

$$L_{j-1} = -\frac{1}{k}\left(w_{j-1} - g_{j-1}\right) + h(x,t) \quad (38)$$

Using equations (36-38) in equation (29), we obtained

$$\frac{1}{h^2}\left(M_{j-1}-2M_j+M_{j+1}\right)=\alpha(\frac{1}{k}\left(w_{j-1}-g_{j-1}\right)+h(x,t))+2\beta(\frac{1}{k}\left(w_j-g_j\right)+h(x,t))+\alpha(\frac{1}{k}\left(w_{j+1}-g_{j+1}\right)+h(x,t)) \quad (39)$$

$$\Longrightarrow M_{j+1}\left(\alpha-\frac{1}{h^2}\right)+2M_j\left(\beta+\frac{1}{h^2}\right)+M_{j-1}\left(\alpha-\frac{1}{h^2}\right)+\alpha\left(\frac{w_{j-1}-g_{j-1}}{k}\right)+2\beta\left(\frac{w_j-g_j}{k}\right)+\alpha\left(\frac{v_{j+1}-g_{j+1}}{k}\right)-2(\alpha+\beta)h(x,t)=0 \quad (40)$$

The equations (35) and (40) form a comprehensive system of algebraic equations. These are associated with the BCs given in equations (3) and (6). Simple technique equations could be used to solve it.

3. Results and Discussion

3.1. Test problem 1

We consider

$$u_{tt}+u_{xxxx}=0$$

having initial conditions as

$u(x,0)=cosx$ and $u_t(x,0)=0$

and exact solution as

$$u(x,t)=cosx.cost$$

Table 1. Comparison of the exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

x	Exact	PCSM	Absolute error for PCSM	NPCSM	Absolute error for NPCSM
0.2	0.809017	0.595175043	0.213841957	0.809016	1E-06
0.4	0.309017	0.198773877	0.110243123	0.309017	0
0.6	-0.309017	-0.198823552	0.110193448	-0.309017	0
0.8	-0.809017	-0.595224671	0.213792329	-0.809016	1E-06

Table 2. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

x	Exact	PCSM	Absolute error for PCSM	NPCSM	Absolute error for NPCSM
0.2	0.804975	0.595175043	2.09800E-01	0.801032	3.94300E-03
0.4	0.307473	0.198773877	1.08699E-01	0.30572	1.75300E-03
0.6	-0.307473	-0.198823552	1.08649E-01	-0.305426	2.04700E-03
0.8	-0.804975	-0.595224671	2.09750E-01	-0.80177	3.20500E-03

Table 3. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

x	Exact	PCSM	Absolute error for PCSM	NPCSM	Absolute error for NPCSM
0.2	0.808977	0.595175043	0.213801957	0.808927	5E-05
0.4	0.309002	0.198773877	0.110228123	0.308987	1.5E-05
0.6	-0.309002	-0.198823552	0.110178448	-0.308987	1.5E-05
0.8	-0.808977	-0.595224671	0.213752329	-0.808927	5E-05

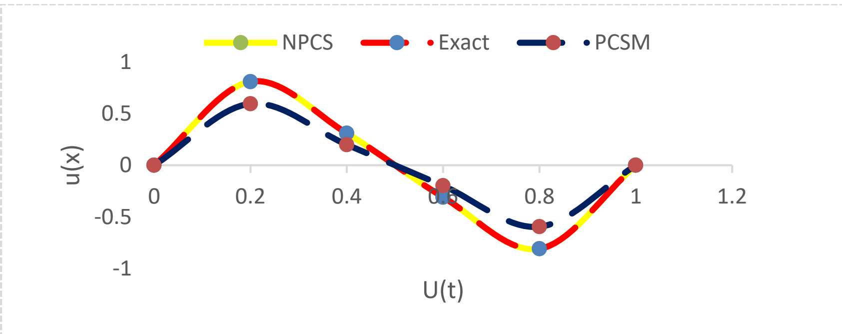

Figure 1. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0. 1

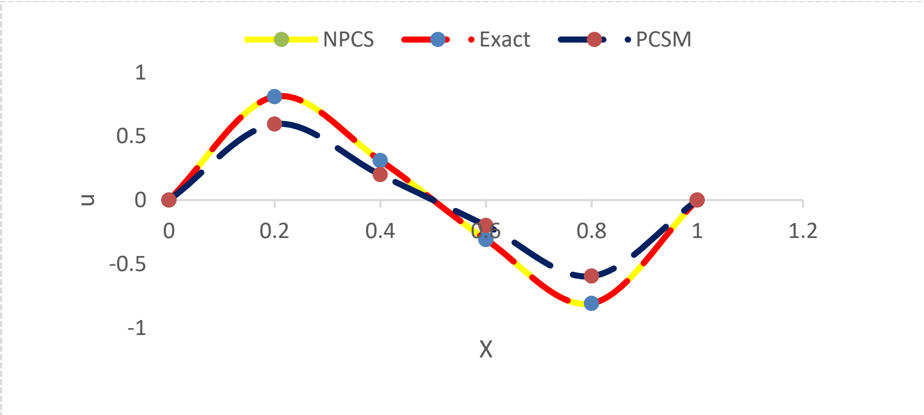

Figure 2. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

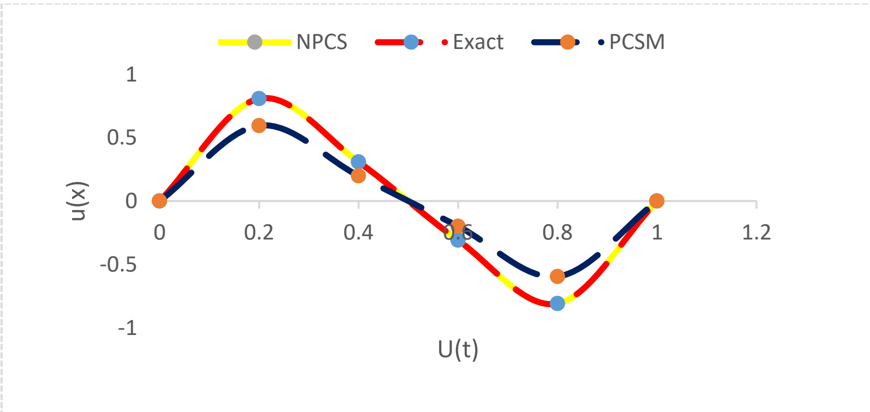

Figure 3. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

Test problem 2

We consider

$$u_{tt} + u_{xxxx} = 0$$

having initial conditions as

$u(x, 0) = 2$ and $u_t(x, 0) = sinx$

and exact solution as

$$u(x, t) = 2 + \text{sinx. sint}$$

Table 4. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

x	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	2.5868061	2.532491594	5.43145E-02	2.058197319	4.83291E-04
0.4	2.949472214	2.861589498	8.78827E-02	2.09416524	7.81981E-04
0.6	2.949472214	2.861589498	8.78827E-02	2.09416524	7.81981E-04
0.8	2.05868061	2.053249159	5.43145E-03	2.058197319	4.83291E-04

Table 5. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

x	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	2.05868061	2.053249159	5.43145E-03	2.005877266	4.88988E-07
0.4	2.094947221	2.08615895	8.78827E-03	2.009509615	7.91199E-07
0.6	2.094947221	2.08615895	0.008788272	2.009509615	7.91199E-07
0.8	2.05868061	2.053249159	5.43145E-03	2.005877266	4.88988E-07

Table 6. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

x	Exact	PCSM	Absolut error for PCSM	NPCSM	Absolute error for NPCSM
0.2	2.005868061	2.005324916	0.000543145	2.000587727	4.88988E-08
0.4	2.009494722	2.008615895	0.000878827	2.000950962	7.91199E-08
0.6	2.009494722	2.008615895	0.000878827	2.000950962	7.91199E-08
0.8	2.005868061	2.005324916	0.000543145	2.000587727	4.88988E-08

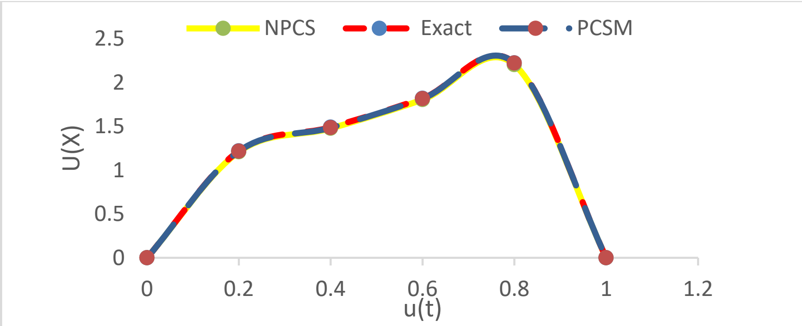

Figure 4. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

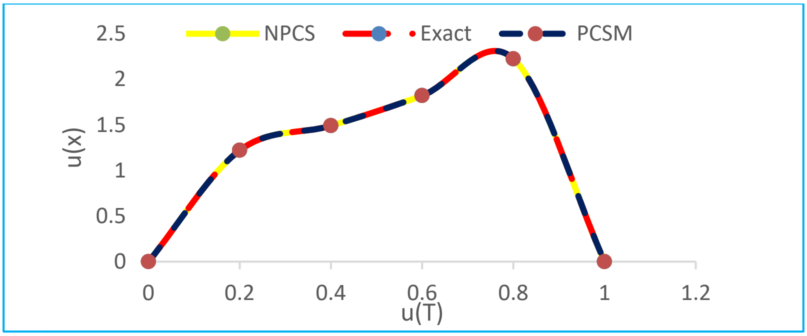

Figure 5. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.01

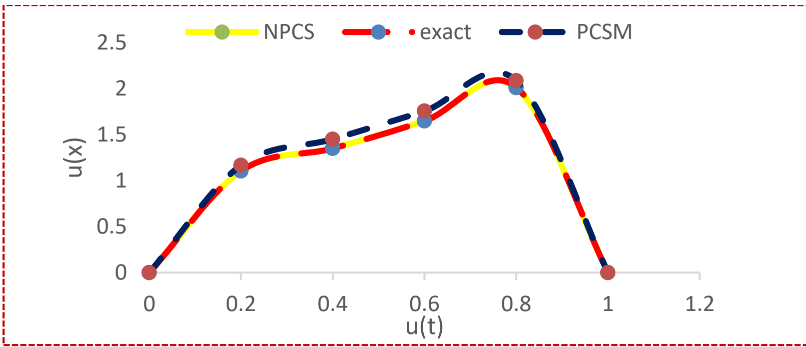

Figure 6. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.001

Test Problem 3:

We consider
$u_{tt} + u_{xxxx} = 0$
having initial conditions as
$u(x,0) = cosx \; and \; u_t(x,o) = \; -sinx$
, boundary conditions as
$u(x,0) = cosx, and \; u(\pi,t) = -cost$
and exact solution as

$$u(x,t) = \cos(x+t)$$

Table 7. *Comparison Of Error*

	Time step	**X=0.2**	**X=0.4**	**X=0.6**	**X=0.8**
Purposed method	0.1	2.74329E-03	2.74329E-03	2.74329E-03	2.74329E-03
	0.01	3.94624E-05	3.94624E-05	3.94624E-05	3.94624E-05
	0.001	4.0032E-07	4.0032E-07	4.0032E-07	4.0032E-07
Comparison [6]	0.1	2.82948E-03	9.5745E-03	1.2616E-02	1.09113E-02
	0.01	2.1013E-05	8.40208E-05	1.14291E-04	1.01192E-04
	0.001	2.101E-07	8.40038E-07	1.14291E-06	1.01912E-06

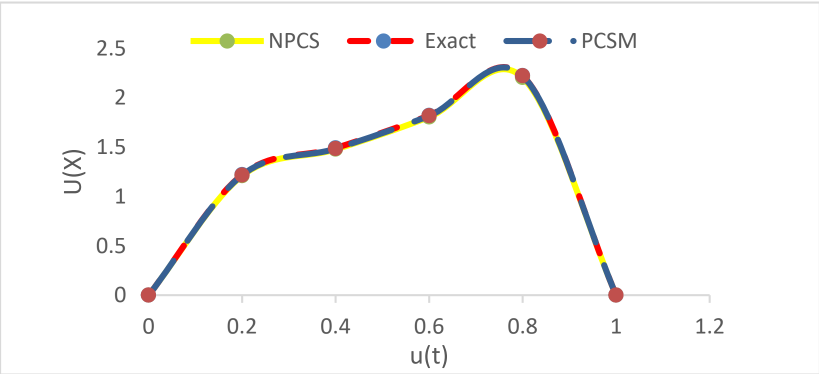

Figure 7. Comparison of exact solution with numerically obtained results by polynomial and non-polynomial cubic spline method at h=1/5 and k=0.1

This study confirmed the validity of the polynomial and non-polynomial cubic spline methods through its application to a variety of test problems. The results were compared with the results of already published research in related literature.

The results of test problem 1 are shown in Table and Figure 1-3. As depicted in Table 3, the minimum absolute error obtained by the non-polynomial cubic spline method (NPCSM) decreased to 10^{-8} at h=$\frac{1}{5}$, and k=0.001. Conversely, the absolute error presented by the polynomial cubic spline method (PCSM) goes to a limit of 10^{-3} at h= $\frac{1}{5}$, and k=0.01. A smaller kind of capacity can be observed in the experimentation of test problem 2. The numerical results by PCSM and NPCSM as well as their absolute error with the exact solution are shown in the Table and Figure (4-6) at h=$\frac{1}{5}$, k=0.1, 0.01, and 0.001. It was also observed that NPCSM performed better than PCSM as shown in Table 4-6. Additionally, a minimum absolute error of 10^{-5} by the NPCSM and 10^{-2} by PCSM was also identified as can be seen in Table 3 and 6, respectively.

Overall, NPCSM was found to be better than PCSM and other already existing methods. NPCSM provided better results for the smaller time steps. A slight improvement was observed in the aforementioned results with the decrease in spatial step size h.

4. Conclusion

This study aimed to develop an interpolation technique for the solution of the fourth order homogeneous parabolic partial differential equations (PDEs). There are several techniques that can be used to solve ordinary differential equations (ODEs) and partial differential equations (PDEs). In this study, the purposed method is the non-polynomial cubic spline method (NPCSM), it was used to solve the fourth order homogeneous parabolic partial differential equations (PDEs). The numerically obtained results were compared with results obtained from the polynomial cubic spline method (PCSM) and other already existing methods [12, 7]. Subsequently, the numerical results were verified at different time steps and spatial intervals by comparing them with the exact solution. The validity of the method was checked through test problems. The superiority of the constructed technique can be seen in test problems (1-2), in which the numerical results obtained by NPCSM are compared with the numerical results obtained using PCSM. This method primarily depends upon the defined NPCSM. If any other NPCSM is utilized, the results can be improved further.

Conflict of Interest

The authors declare no conflict of interest.

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