Analytical solutions of fractional partial differential equations for the second grade fluid flow

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Abstract

This research work is related to unsteady movement of second-grade fluid over an infinite plate. The governing equations for flow are developed through constitutive relations. Then classical model extended to fractional order model with power law fractional differential operator. The Laplace transform (LT) technique is applied to find the analytical results and stated as series satisfy the boundary conditions. To see physical significance of flow parameters some graphs are displayed. Recent results from the existing literature are recovered to validate.

Key words: CPC fractional derivative; Fractional model; Analytical solutions; Laplace transform method.

1 Introduction

Unlike Newtonian fluids, non-Newtonian fluids cannot be reported as simply. Because of the intricacy and complexity in the nature of non-Newtonian fluids, many models and constitutive equations have been suggested. [1]. Honey, ketchup, blood, shampoo, greases and certain oils falls under the category of non-Newtonian fluids [3]. These fluids are not just bound to physics and engineering but the non-linear comportment of these fluids manifest them in many other fields like bio-engineering, electrochemistry, biophysics, rheology, viscoelasticity and drilling operations etc. [4] The second-grade liquids, which structure a subclass of the differential liquids have been concentrated effectively in different types of flows [2].

There are very few cases in which Navier Stoke's equations can be analyzed. The beginning purpose of the fractional model of a viscoelastic fluid is generally an old-style differential condition which is changed by supplanting the time subordinate of a whole number request by the purported Riemann-Liouville fractional calculus operator [11]. Because of expanded enthusiasm for modeling with the assistance of the fractional approach, a few fluid models are summed up and fractional

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models have been created [6].

Since last few decades, this field has been catching attentions of many researchers. Nadeem et al. presented Caputo-Fabrizio (CF) fractional operator to MHD stream of a second-grade liquid combined with radioactive heat transmission [5]. Nehad and Ilyas presented an approach of (CF) fractional operator to the thermal investigation of a second-grade liquid where it is applied on an in-filled flat plate this is vertical [6]. Tan et al. presented the introduction of fractional calculus into the rheological constitutive representation of a generalized second-grade liquid [7]. Fan et al. explored the Rayleigh Stoke's problem for second-grade liquid subject. Fractional calculus approach was chosen for describing the constitutive relationship of the model. They used Fourier sine transform and the fractional (LT) to obtain the analytical results of the velocity and temperature [9]. Meherdad evaluated the unsteady pulsatile blood movement in an artery which also included the impacts of body acceleration. Fahraeus Lindqvist effect was applied on the model and equations were solved numerically, making it dimensionless first [10]. Khan et al. considered and investigated the effects of a porous medium and on a number of second grade unidirectional flows. Their research also claims that the MHD movements were made by using the concepts of periodic pressure gradient and by the impulsive motion of one and more than one boundaries or by an oscillating plate [12]. Waqas et al. envisioned the progression of second-grade nano-fluid with heat, motile micro-organisms, and mass exchange rates overextending surface. The adjusted second-grade fluids used to break down the rheological conduct [13]. Ghadikolaei et al. investigated the stream and warmth move of an incompressible homogeneous second-grade liquid over an extending sheet channel. Hemotopy Perturbation strategy is utilized to fathom nonlinear differential conditions in the paper [14]. Massoudi and Phuoc proposed a changed constitutive condition for a second grade liquid with the goal that the model would be appropriate for considers where shear-thickening may happen [15]. Qi and Xu examined a temperamental channel stream of a viscoelastic fluid with the fractional Maxwell model [16]. Ibrahim presented the consolidated impacts of the induced magnetic field and convective warmth move in Maxwell nanofluid in the area of stagnation point [17]. Ikram et al. [18] discussed MHD movement of a Newtonian liquid in symmetric channel with ABC fractional model having hybrid nanoparticles.

Recently, Baleanu et al. [19] introduced constant proportional Caputo (CPC) fractional operator by combining Riemann Liouville integral and Caputo fractional operator. Imran et al. [20] studied fluid flow for viscous fluid for such geometry via (CPC) fractional operator. Further Imran et al. [21] discussed MHD effects and find analytical solutions with (LT) method. The heat transfer of viscous nanofluid with (ABC) and (CPC) fractional operators has been analyzed by [22, 23]. Ikram et al. [24] evaluated the Brinkman type fluid (BTF) non-integral model containing hybrid nanoparticles through (CPC) fractional derivative. Chu et al. [25] presented an analysis of non-integral model of viscous nanofluid carrying hybrid nanoparticles, copper (Cu) and aluminium oxide (Al_2O_3) with base fluid water (H_2O) under the MHD effect in a microchannel via (CPC) fractional derivative. Therefore, our prime interest to extend this problem to non-Newtonian fluid called second grade fluid with recently introduced fractional operator (CPC). The main benefit of this operator is that it is suitable to exhibit the strong memory effect of the studied problem.

2 Preliminaries

Constant Proportional-Caputo Fractional Operator:
CPC fractional operator of order $\gamma \in [0,1)$ is described as [19]

$$^{CPC}D_t^{\gamma} g(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \left(L_1(\gamma) g(x) + L_0(\gamma) g'(x)\right) (t-x)^{-\gamma} dx.$$

where g is differentiable and both g and g' are locally L^1 functions on the positive reals and $L_0(\gamma)$, $L_1(\gamma)$ lie between $[0,1]$.

The LT of CPC fractional operator is

$$L\left[^{CPC}D_t^{\gamma} g(t)\right] = \left[\frac{L_1(\gamma)}{s} + L_0(\gamma)\right] s^{\gamma} \bar{g}(s) - L_0(\gamma) s^{\gamma-1} g(0)$$

3 Constitutive Equations

The constitutive equations of second grade can be written as [8]

$$\mathbf{T} = \mu \mathbf{C}_1 + \alpha_1 \mathbf{C}_2 + \alpha_2 \mathbf{C}_1^2 - p\mathbf{I} \tag{1}$$

where $\mathbf{T}$ denotes Cauchy stress tensor, μ is co-efficient of viscosity, $p\mathbf{I}$ is intermediate spherical stress. Similarly, α_1 and α_2 represents normal stress moduli. Likewise, $\mathbf{C}_1$ and $\mathbf{C}_2$ represents kinematic tensors which can be defined as

$$\mathbf{C}_1 = [\bigtriangledown \mathbf{V}] + [\bigtriangledown \mathbf{V}]^T \tag{2}$$

$$\mathbf{C}_2 = \mathbf{C}_1[\bigtriangledown \mathbf{V}] + [\bigtriangledown \mathbf{V}]^T \mathbf{C}_1 + \frac{d\mathbf{C}_1}{dt} \tag{3}$$

In Eq. (2) and Eq. (3), $\mathbf{V}$ represents the velocity, $\bigtriangledown$ represents the gradient and $\frac{d}{dt}$ shows material time derivative.
As we know that the liquid is incompressible so it will experience isochoric motion only. This can be written as

$$\bigtriangledown . \mathbf{V} = 0 \tag{4}$$

Equation of motion becomes

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{F} + \bigtriangledown \mathbf{T} \tag{5}$$

where $\mathbf{F}$ is the body force and ρ is the density of the fluid. In this research, we will be considering the model represented in Eq. (1). This model must fulfil the some conditions in order to be well-suited or compatible with the laws of thermodynamics. These conditions are:[8]

$$\alpha_1 + \alpha_2 = 0, \quad \mu \geq 0, \quad \alpha_1 \geq 0$$

4 Mathematical modeling

Suppose the unsteady movement of an incompressible second-grade liquid on a flat (solid) plate with the following assumptions.
(i) The plate fill up the half space $y > 0$ taking xz-plane.
(ii) At start, velocities of both liquid and plate are supposed to zero.
(iii) Later, plate moves with velocity U (constant).
(iv) The movement happens on x-axis.
(v) Pressure gradient and body force are absent in the movement.
(vi) The incompressibility constraints fulfil for the velocity field of the form $V = (u, 0, 0)$.
By the law of conservation of momentum, governing equation for velocity is as

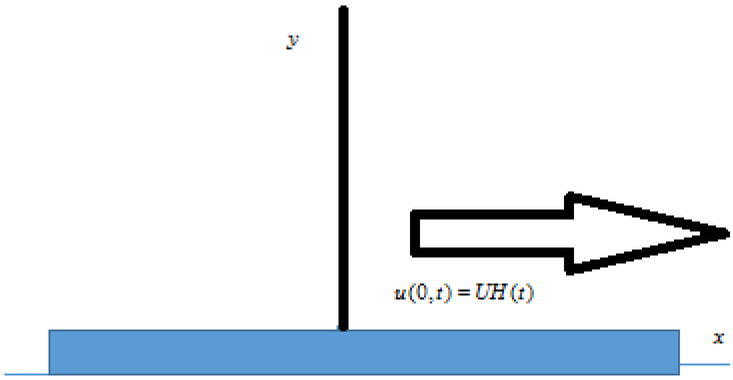

Figure 1: Physical model

$$\rho u_t(y,t) = (\mu + \alpha_1 \frac{\partial}{\partial t}) u_{yy}(y,t), \tag{6}$$

subject to the following constraints:

$$u(y,0) = 0 \quad \forall\ y, \quad u(0,t) = UH(t), \quad for\ t > 0,$$

$$u(y,t) \to 0,\ as\ \ y \to \infty. \tag{7}$$

Initiating the dimensionless variables,

$$\Psi = \frac{u}{U}, \quad \xi = \frac{y}{L}, \quad \tau = \frac{U}{L} t,$$

in Eqs. (6) and (7), we have,

$$\mathrm{Re}\Psi_\tau(\xi,\tau) = (1 + \alpha_2 \frac{\partial}{\partial \tau}) \Psi_{\xi\xi}(\xi,\tau), \tag{8}$$

with constraints

$$\Psi(\xi,0)=0, \qquad \Psi(0,\tau)=1, \qquad for \quad \tau>0,$$
$$\Psi(\xi,\tau)\to 0, \qquad as \qquad \xi\to\infty. \tag{9}$$

where $\mathrm{Re}=\frac{\rho UL}{\mu}$ is Reynolds number (dimensionless) and $\alpha_2=\frac{\alpha_1 U}{\mu L}$ second grade parameter.

5 Constant proportional fractional initial boundary value problem

CPC fractional model of Eq. (8) is as

$$\mathrm{Re}\Psi_\tau(\xi,\tau)=\left(1+\alpha_2{}^{CPC}D_\tau^\alpha\right)\Psi_{\xi\xi}(\xi,\tau), \tag{10}$$

initial and boundary constraints are:

$$\Psi(\xi,0)=0, \qquad \Psi(0,\tau)=1, \qquad \psi(\xi,\tau)\to 0 \quad as \quad \xi\to\infty. \tag{11}$$

Using LT on Eqs. (10) and (11), we obtain

$$\mathrm{Re}q\bar{\Psi}(\xi,q)=\left[1+\alpha_2\left(\frac{L_1(\alpha)}{q}+L_0(\alpha)\right)q^\alpha\right]\bar{\Psi}_{\xi\xi}(\xi,q), \tag{12}$$

$$\bar{\Psi}(0,q)=\frac{1}{q}, \qquad \bar{\Psi}(\xi,q)\to 0, \quad as \quad \xi\to\infty. \tag{13}$$

where q is LT parameter.
The solution of Eq. (12) subject to Eq. (13) is given by

$$\bar{\Psi}(\xi,q)=\frac{1}{q}+\sum_{\eta_1=1}^{\infty}\sum_{\eta_2=0}^{\infty}\sum_{\eta_3=0}^{\infty}\frac{\left(-\xi\sqrt{\mathrm{Re}}\right)^{\eta_1}(-\alpha_2)^{\eta_2}\left[L_1(\alpha)\right]^{\eta_3}\Gamma(\frac{\eta_1}{2}+\eta_2)\Gamma(\eta_2+1)}{\eta_1!\eta_2!\eta_3!\left[L_0(\alpha)\right]^{\eta_3-\eta_2}q^{1-\frac{\eta_1}{2}-\alpha\eta_2+\eta_3}\Gamma(\frac{\eta_1}{2})\Gamma(\eta_2-\eta_3+1)}. \tag{14}$$

Using inverse LT to Eq. (14), we get

$$\Psi(\xi,\tau)=1+\sum_{\eta_1=1}^{\infty}\sum_{\eta_2=0}^{\infty}\sum_{\eta_3=0}^{\infty}\frac{\left(-\xi\sqrt{\mathrm{Re}}\right)^{\eta_1}(-\alpha_2)^{\eta_2}\left[L_1(\alpha)\right]^{\eta_3}\tau^{-\frac{\eta_1}{2}-\alpha\eta_2+\eta_3}\Gamma(\frac{\eta_1}{2}+\eta_2)\Gamma(\eta_2+1)}{\eta_1!\eta_2!\eta_3!\left[L_0(\alpha)\right]^{\eta_3-\eta_2}\Gamma(1-\frac{\eta_1}{2}-\alpha\eta_2+\eta_3)\Gamma(\frac{\eta_1}{2})\Gamma(\eta_2-\eta_3+1)}. \tag{15}$$

6 Solution of second grade fluid with Caputo

The result of problem stated in Eqs. (8) and (9) with Caputo fractional derivative with the application of LT is as

$$\Psi(\xi,\tau)=1+\sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty}\frac{\left(-\xi\sqrt{\mathrm{Re}}\right)^{j_1}(-\alpha_2)^{j_2}\tau^{-\frac{j_1}{2}-\alpha j_2\Gamma(\frac{j_1}{2}+j_2)}}{j_1!j_2!\Gamma(1-\frac{j_1}{2}-\alpha j_2)\Gamma(\frac{j_1}{2})}. \tag{16}$$

7 Solution of second grade fluid with Caputo-Fabrizio

The result of problem specified in Eqs. (8) and (9) with Caputo-Fabrizio fractional derivative with the application of LT is as

$$\Psi(\xi,\tau)=1+\sum_{\varpi_1=1}^{\infty}\sum_{\varpi_2=0}^{\infty}\sum_{\varpi_3=0}^{\infty}\frac{\left(-\xi\sqrt{\mathrm{Re}}\right)^{\varpi_1}(-\alpha_2)^{\varpi_2}(\chi)^{\frac{\varpi_1}{2}+\varpi_2}(\alpha\chi)^{\varpi_3}\tau^{\varpi_3}\Gamma(\frac{\varpi_1}{2}+\varpi_2)\Gamma(\frac{\varpi_1}{2}+\varpi_2+1)}{\varpi_1!\varpi_2!\varpi_3!\Gamma(1+\varpi_3)\Gamma(\frac{\varpi_1}{2})\Gamma(\frac{\varpi_1}{2}+\varpi_2-\varpi_3+1)} \tag{17}$$

where $\chi=\frac{1}{1-\alpha}$.

8 Numerical results and discussion

The effect of Re on velocity field can be seen in fig. 2 and noticed that velocity is reduced by Re. It is because of that Re is ratio of inertial forces to viscous forces, flow of fluid reduced. Fig. 3 indicates the impact of α_2 on velocity distribution and shows that by increasing value of α_2 velocity decreases. Figures 4-5 are designed to see the influence of α on velocity for little and big values of time. For small time, it is noted from Fig. 4 velocity is decreasing function close to the plate for large values of α. This quick decay in velocity is because of rise in temperature and velocity boundary layer for rising α, while Fig. 5 depicts for big time the α shows the opposite behavior than for small time t. In such a way, α appears double action for this flow problem with little and big time.

The comparison of velocity field with C, CF and CPC fractional derivatives by fixing other parameters with changing α are seen in Fig. 6-9. By these figures,we concluded that velocity profile with CPC fractional operator displays more decline than other all fractional approaches. Since CPC operator based on Riemann Liuovile and Caputo fractional operators. So, it is resulted that CPC operator that is applied in this effort is better to show the decline of velocity rather than others. And also noted that, by rising values of α, thermal conductivity decreases.

Fig. 10 is designed for comparison of velocities with viscous fluid and second-grade liquid and concluded that second-grade liquid flow is slow than viscous fluid flow. If $\alpha_2=0$, then second-grade liquid and viscous fluid flow shows same behavior as shown in fig. 11. Table 1 shows the effect of α on second-grade liquid flow and viscous fluid flow and concluded that second grade fluid flow is slow than viscous fluid flow.

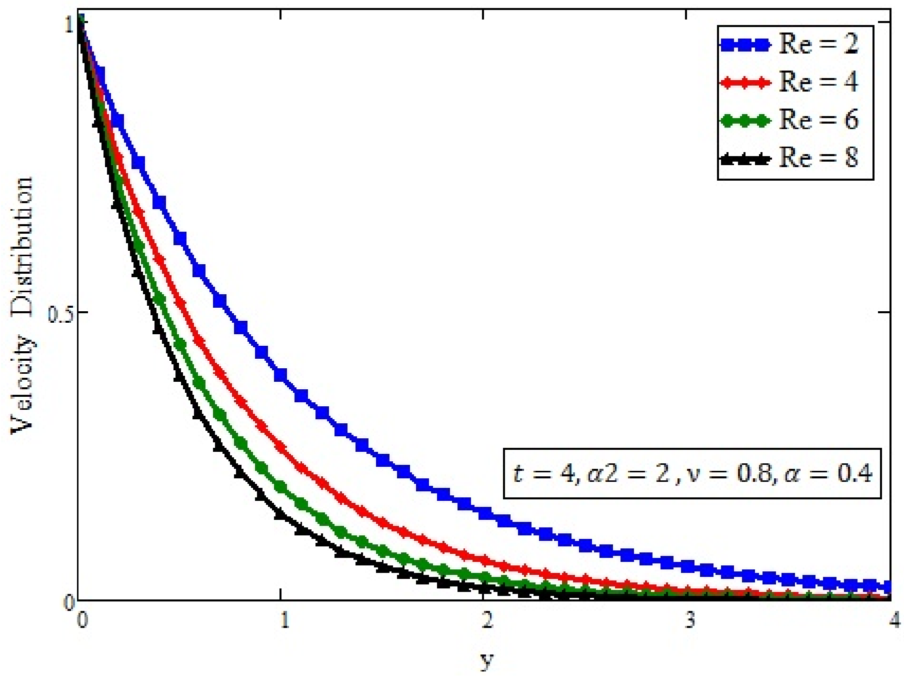

Figure 2: The effect of Re on velocity field against y

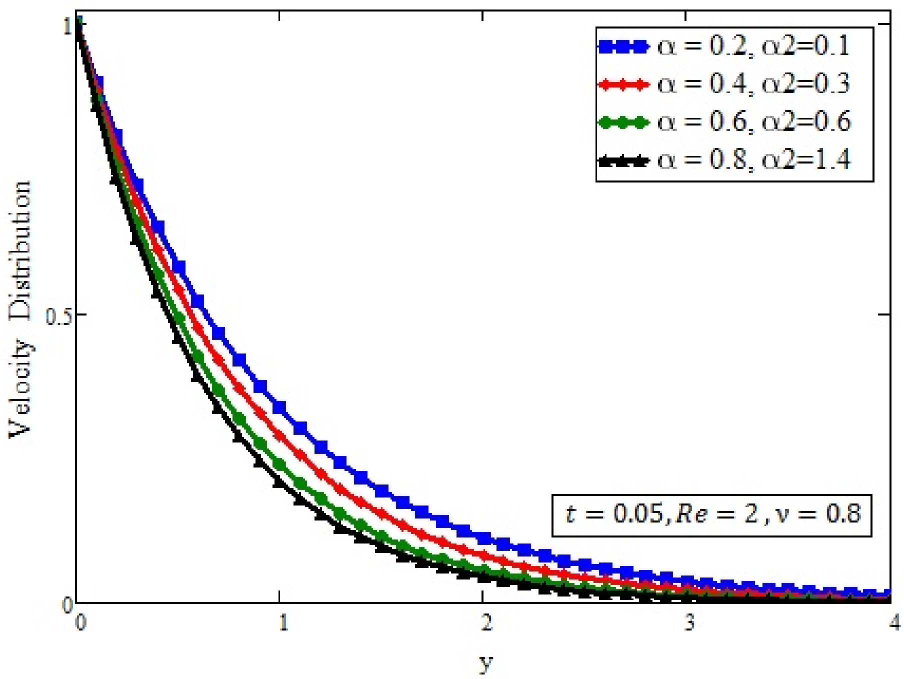

Figure 3: The impact of α_2 on velocity field against y

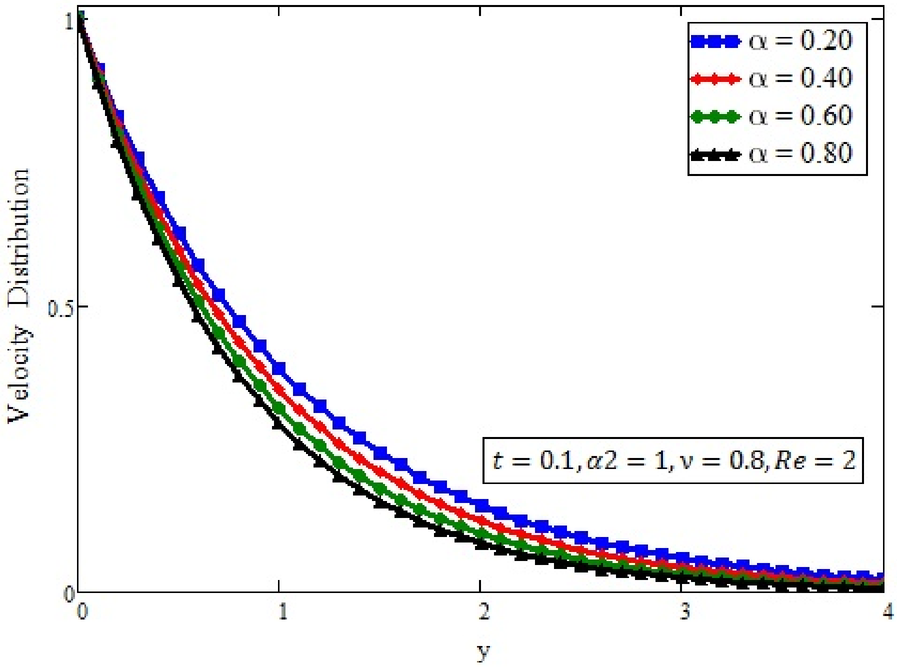

Figure 4: The effect of α on velocity field against y for small time

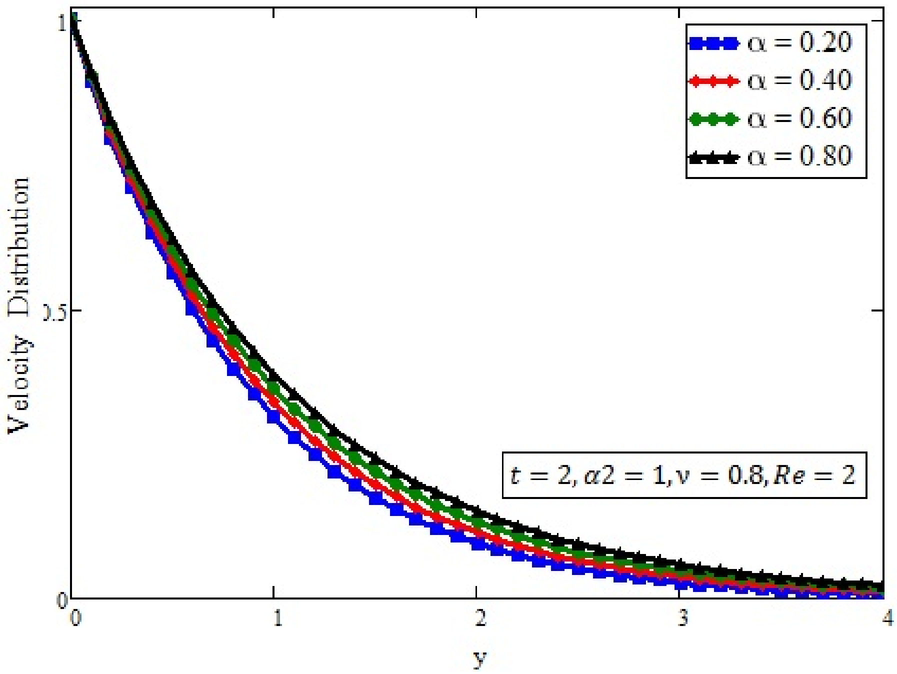

Figure 5: The effect of α on velocity field against y for large time

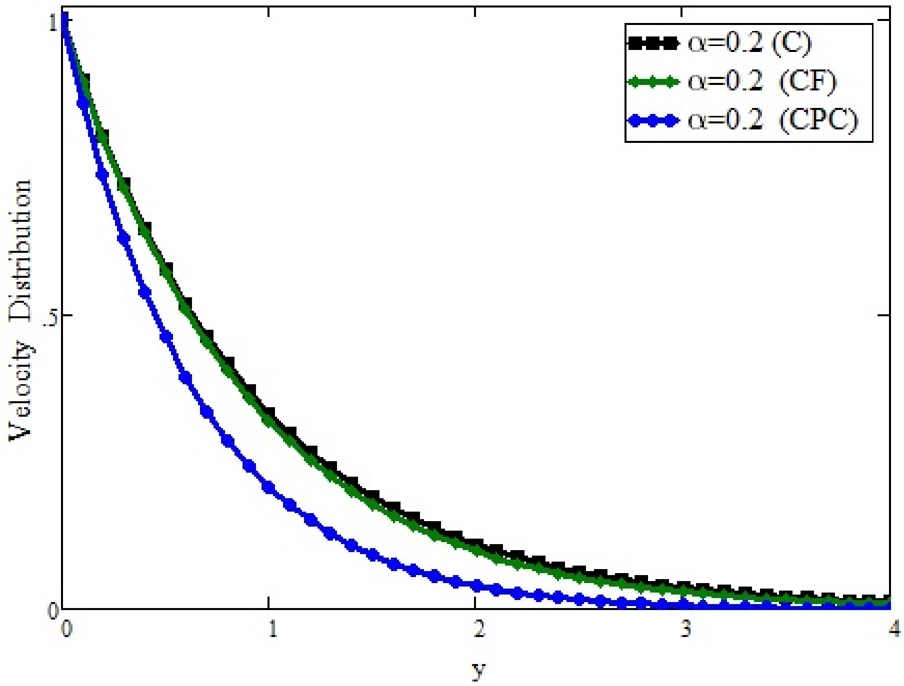

Figure 6: Differentiation between the velocities with CPC, C, CF

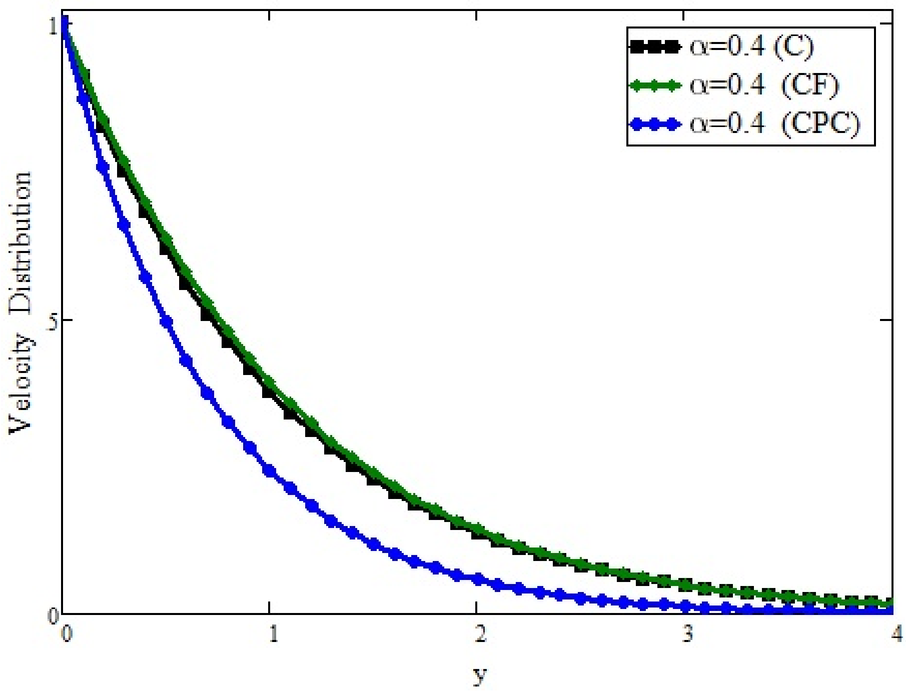

Figure 7: Differentiation between the velocities with CPC, C, CF

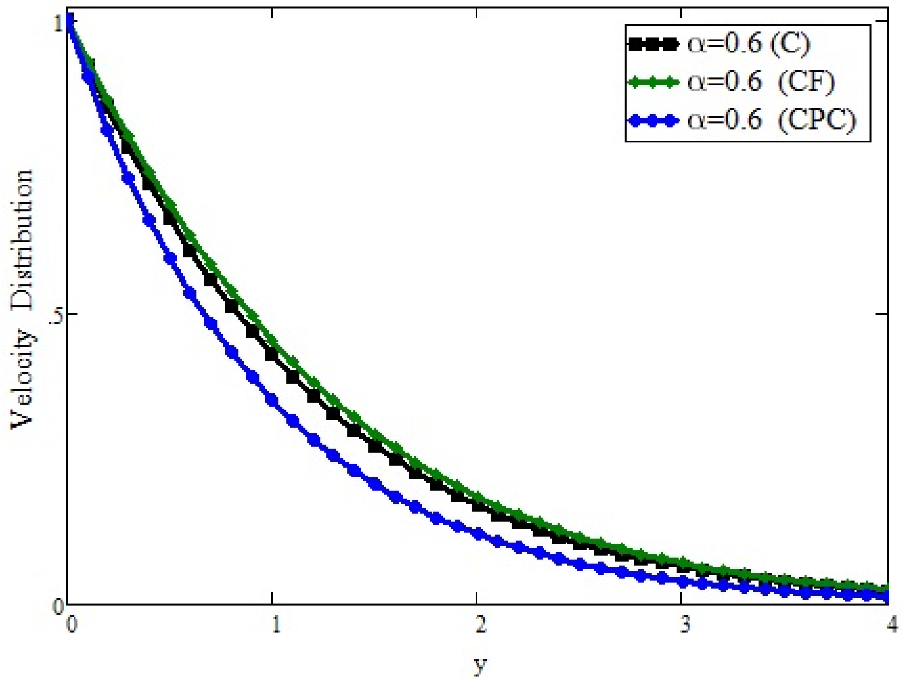

Figure 8: Differentiation between the velocities with CPC, C, CF

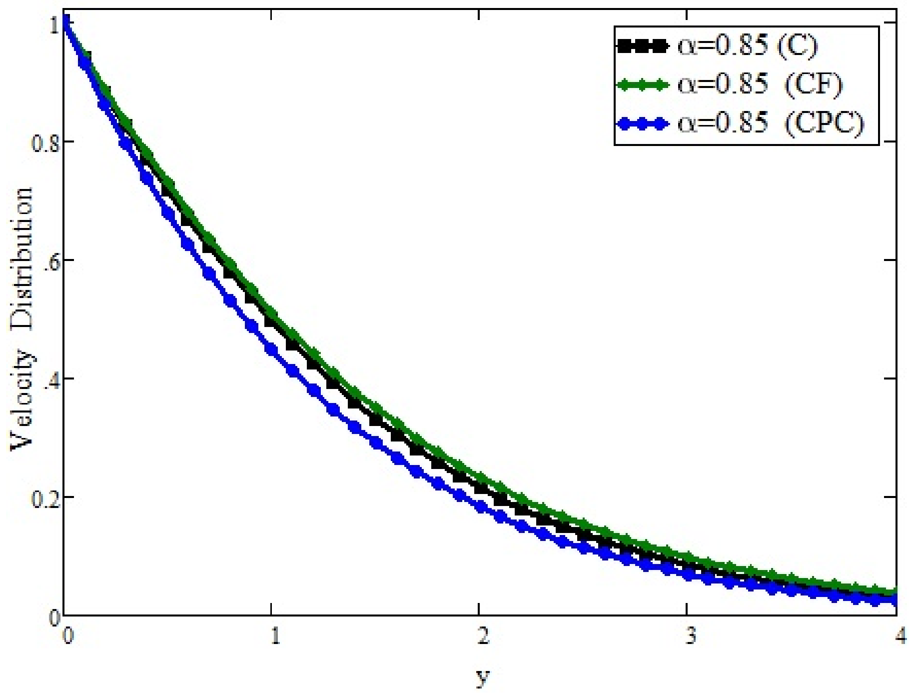

Figure 9: Differentiation between the velocities with CPC, C, CF

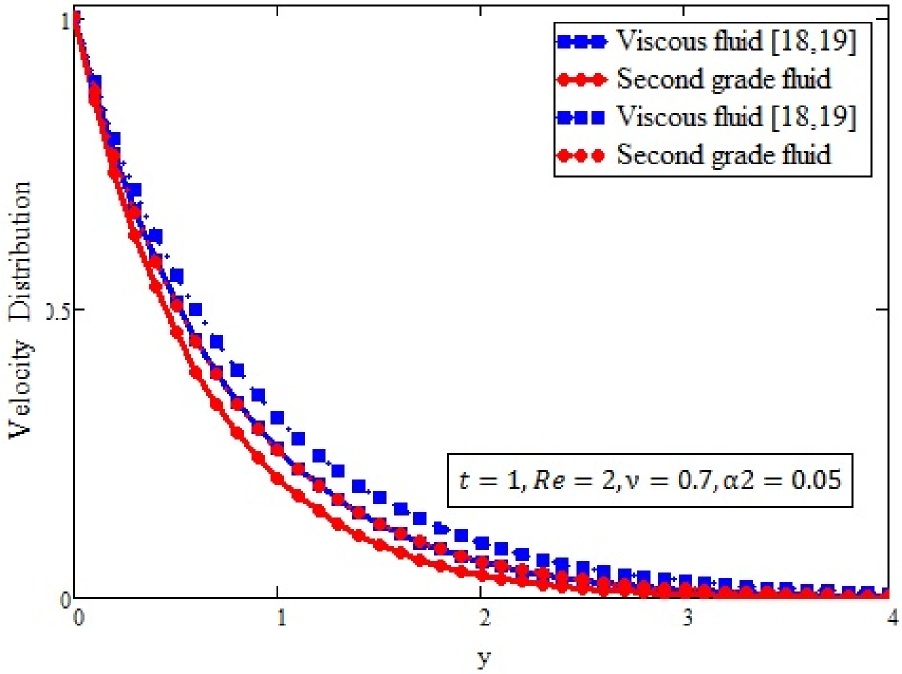

Figure 10: Differentiation between the velocities with Viscous fluid [18,19] and second grade fluid

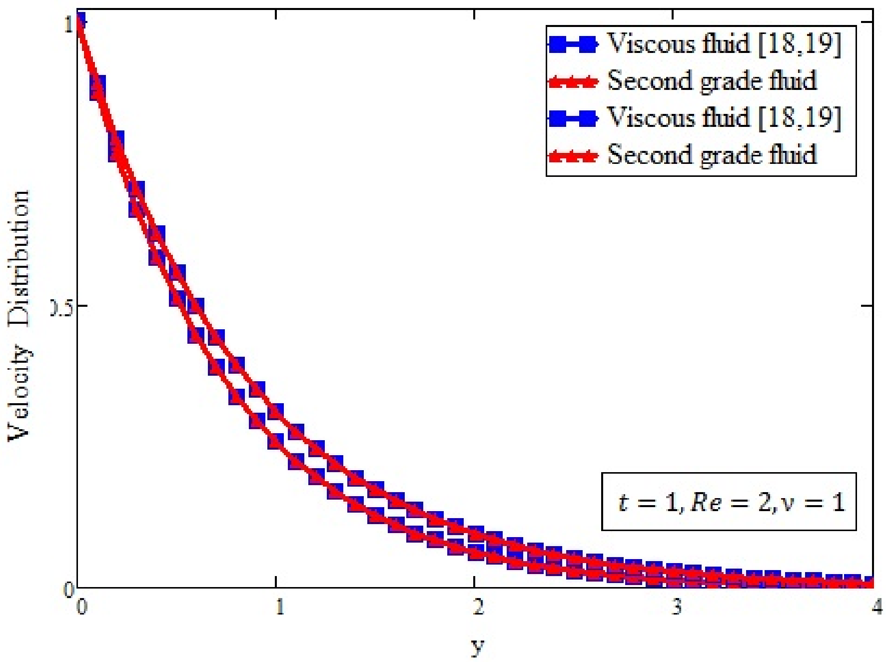

Figure 11: Differentiation between the velocities with Viscous fluid [18,19] and second grade fluid when $\alpha_2 = 0$

Table 1: The effect of fractional parameter α

α	Viscous Fluid [18,19]	Second grade Fluid Present
0	0.581	0.531
0.1	0.587	0.538
0.2	0.594	0.546
0.3	0.601	0.554
0.4	0.609	0.562
0.5	0.617	0.571
0.6	0.626	0.580
0.7	0.635	0.589
0.8	0.645	0.599
0.9	0.656	0.610
1.0	0.667	0.621

9 Conclusions

This research work is related to unsteady flow of second-grade fluid over an infinite plate. The governing equations for flow are developed through constitutive relations. Then classical model extended to fractional order model with power law fractional differential operator. The Laplace transform (LT) technique is applied to find the analytical results and stated as series satisfy the boundary conditions. To see physical significance of flow parameters some graphs are displayed. Recent results from the existing literature are recovered to validate.

The key points of the present work are:

• By rising the value of time, the density of velocity boundary layer increases with all the non-integer models but for rising $\alpha \to 1$ it reduced.

• In a contrast between CPC and other fractional operators, we agreed that CPC fractional shows better decline of the velocity than them.

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