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Solutions of Volterra Integral Equations (VIEs) of the Second Kind with Bulge Function using Aboodh Transform

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ABSTRACT

A large class of complexities in mathematical physics, applied mathematics, and engineering are expressed as differential equations with few additions and certain conditions. This research focuses on the solution of Volterra integral equations (VIEs) of the second kind, with bulge functions as known functions. To obtain an analytical solution, the Aboodh transform, the Aboodh inverse transform, and the convolution theorem are employed, since it is required to discover the precise solution of VIEs. This solution is compared with a numerical solution using the modified Simpson method. Finally, it is represented graphically.

Keywords: Aboodh Transformation Method (ATM), bulge function, convolution theorem, improved Simpsons method, Taylor series expansion, \Volterra integral equations (VIEs)

INTRODUCTION

In this research, the Aboodh transformation method [1, 2] is used to solve problems without making use of a new frequency range. Also, the emphasis remains on solving the Volterra integral equations (VIEs) [3, 4] commonly applied in many scientific research areas, such as epidemic distribution, semiconductor devices, and population dynamics. VIEs appear in many physical applications, such as in industries to estimate increasing and decreasing productivity, and in engineering applications, such as general heat transfer and diffusion processes [5]. Recently, several researchers investigated the solutions to these problems. For instance, Kamyad et al. [6] proposed an interpolation and estimation method for VIEs.

Ali et al. [7] studied the non-integer order partial differential equation (PDE) by using non-integer power series technique. Via the application of the direct power series method, the use of fractional-order power series technique for calculating the non-linear fractional order PDEs was found to be comparatively simple in application. Hence, it was concluded that non-integer power series techniques are very effective in constructing fractional power series solutions for fractional PDEs of any order, time, and space which reduces the computational cost of solving such problems to a great extent. Mirzaee [8] proposed a numerical method for solving the second type of Volterra linear integral equation based on the Simpson adaptive square method.

Rahman et al. [9] used the famous Galerkin-weighted residual method to numerically solve the second type of VIEs with regular kernels. They used Chebyshev polynomial as an experimental function to obtain a simple and efficient matrix formula. Haarsa and Pothat [10] proposed a solution for the varied behavior of the bulge function in the VIEs of the second type using Laplace transform. Integral equations [11] are imperative in many applications.

Zia et al. [12] presented an algorithm for solving the second-kind fuzzy fractional linear Volterra integro-differential equation (VIDE) with a separable kernel using generalized Hukuhara differentiability and the Laplace-Adomian decomposition method (LADM). By comparing the approximate fuzzy solution with the exact solution through numerical simulation results, it was found that LADM is easier and more accurate and can produce efficient results with fast convergence.

Hassan et al. [13] studied the existence and uniqueness of the solutions of the linear systems of mixed Volterra-Fredholm integral equations of the second kind in complete metric spaces. Analytical solution was obtained using Banach's theorem and the approximate solution of the fixed point method. Comparisons were made to determine the accuracy of the results and the efficiency of the methods.

Shah et al. [14] analyzed the solution of the regularized fractional long wave (RLW) equation by using Elzaki transform of the Caputo fractional derivative and the Atangana-Baleanu fractional derivative. In order to ensure the effectiveness of the proposed technique, differences in the solutions for the Caputo and Atangana-Baleanu fractional derivatives were also observed.

Akgul [15] discussed kernel Hilbert space methods for studying fractional differential equations, including Atangana-Baleanu fractional derivatives with non-local and non-singular kernels.

Liu et al. [16] used the Yang homotopy perturbation transform method (YHPTM) to study the solutions of time fractional Klein-Gordon equations in both linear and nonlinear cases, where the fractional derivatives were obtained in the Caputo-Fabrizio. The results of YHPTM proved to be simple but powerful mathematical tools that provide the solutions.

Atangana and Akgul [17] solved the nonlinear transfer function using the fractal Laplace transform with a power-law kernel, an exponential decay kernel, and a generalized Mittag-Leffler kernel. They used Newtonian polynomials to demonstrate the effectiveness of the fractal Laplace transform technique.

Ali and Bhatti [18] carried out a numerical comparison between the Aboodh transformation method (ATM), the differential transformation method (DTM). The solution obtained by DTM converged faster than ATM. These methods can reduce the amount of calculation and work efficiently. Comparing the results of the two transformations, it was found that DTM converges to the true solution faster than ATM.

Ali et al. [19] analyzed the influence of the Adomian decomposition technique of the Aboodh transform on the solution of non-linear Volterra integrals and integro-differential equations based on the Newton-Raphson method. The Newtȯn-Raphsȯn method was used as a term in the Adomian polynomial, which showed the effectiveness of the Aboodh-Adomian decomposition method (AADM) for solving VIDEs. It was observed that the Adomian polynomial technique was slightly modified using the Newton-Raphson method.

Kalim et al. [20] discussed the numerical simulation of the bubble growth of boiling fluid in the VOF model and the effect of sub-cooling and superheating on the bubble. It was found that growth rate and the dimension of bubbles are affected by the local temperature, as the mass of bubbles increases or decreases due to condensation and evaporation. Also, the direction of bubbles does not change and the bubbles face obstacles during movement, depending on their volume friction and vapour velocity in the super-cooled and superheated areas. All of these factors caused the phase change development to which heat transfer method was applied.

Ozdemir et al. [21] investigated the linear volterra integral equation of the first kind using ZZ-Transform. Kotan and Celik [22] used Aboodh transform and compared their solution with the previously cited work. They found an exact solution obtained with very little computational effort and in a very short time. Gecmen et al. [23] proposed arrived at the result of the numerical solution of VIE using the Hosoya polynomial and compared the solutions obtained by the Hosoya method using graphs and tables. MATLAB was used to get results, tables, and error analysis. Numerical solutions were obtained by using the improved Simpson method.

In this article, we studied the second type of VIEs with bulge function. The solution was obtained using the Aboodh transform, the Aboodh inverse transform, the convolution theorem, and the Taylor series expansion.

2.FUNDAMENTAL CONCEPTS

Definition 2.1. Let $\mathrm{f(t)}$ be a function such that $\mathrm{t} \geq 0$, f is of exponential order if there exists the number v, $\mathrm{M} > 0$, and $\mathrm{T} > 0$,

so that

$|f(t)| < Me^{-vt},\ for\ t > T,$

where M must be a finite number.

For a given function, Aboodh transform is defined as follows:

$$\mathcal{A}[\mathrm{f(t)}] = \mathrm{k(v)} = \frac{1}{\mathrm{v}}\int_0^\infty \mathrm{e^{-vt} f(t) dt}, \quad \text{for} \quad t \geq 0 \tag{1}$$

where $\mathrm{f(t)}$ is the real-valued function and v is any real variable, while $k_1 \leq v \leq k_{2,}$ where $\mathrm{k_1}$, $\mathrm{k_2}$ be constants that may be finite or infinite.

The original function is called the inverse Aboodh transform *of* $k(v)$ and is denoted by

$\mathcal{A}^{-1}[\mathrm{k(v)}]$ i.e. $\mathrm{f(t)} = \mathcal{A}^{-1}[\mathrm{k(v)}]$.

Theorem 2.1. The convolution theorem

Let $\mathrm{f(t)}$ and $\mathrm{g(t)}$ be two functions, then

$$\mathcal{A}[(\mathrm{f} * \mathrm{g})(\mathrm{t})] = \mathcal{A}\left[\int_0^\mathrm{t} \mathrm{f}(\mathcal{T})\mathrm{g}(\mathrm{t}-\mathcal{T})\, \mathrm{d}\mathcal{T}\right] = vF(v)G(v) \tag{2}$$

The VIE is a special kind of integral equation. One type has the form

$$y(t) = f(t) + \int_0^t k(t-\mathcal{T})y(\mathcal{T})\, \mathrm{d}\mathcal{T}, \tag{3}$$

where f(t) and $k(t-\mathcal{T})$ are known functions and $y(t)$ is an unknown function that is to be determined. For this paper, we take $f(t)$ as a bulge function [24], that is, $e^{\frac{-(t-l)^2}{2}}$, where "l" is a positive constant.

The improved Simpson method to solve integrals $\int_{x_i+0h}^{x_{i+2h}} f(x)dx$ is as follows:

$$\int_{x_i}^{x_{i+2h}} f(x)dx=\frac{h}{3}[f_i+4f_{i+h}+f_{i+2h}]+\frac{h^4}{180}[f_i'''-f_{i+2h}''']-\frac{h^7}{120}[f^6(\varepsilon_i)], \tag{4}$$

where f^6 represents the sixth derivative of f and $\varepsilon_i \in [x_i, x_{i+2h}]$.

3. VIE SOLUTION OF THE SECOND KIND USING ABOODH TRANSFORM

Theorem 3.1. Aboodh transform of the bulge function $e^{\frac{-(t-l)^2}{2}}$, where "l" is a positive constant. It is expressed by

$$\mathcal{A}\left[e^{\frac{-(t-l)^2}{2}}\right]=e^{\frac{-l^2}{2}\left[\frac{1}{v^2}+\frac{l}{v^3}+\frac{(-1+l^2)}{v^4}+\frac{(-3+l^3)}{v^5}\right]}. \tag{5}$$

Proof: The Taylor series expansion e^x is of the form

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \tag{6}$$

Therefore, by substituting equation (6) with $x=\frac{-(t-l)^2}{2}$, we obtain

$$e^{\frac{-(t-l)^2}{2}}=e^{\frac{-l^2}{2}}+e^{\frac{-l^2}{2}}lt+e^{\frac{-l^2}{2}}\left(-\frac{1}{2}+\frac{l^2}{2}\right)t^2+e^{\frac{-l^2}{2}}\left(-\frac{l}{2}+\frac{l^3}{6}\right)t^3. \tag{7}$$

By taking the Aboodh transformation method (ATM) in (7) and the linearity of Aboodh we conclude

$$\mathcal{A}\left[e^{-\frac{(t-l)^2}{2}}\right]=e^{\frac{-l^2}{2}\left[\frac{1}{v^2}+\frac{l}{v^3}+\frac{(-1+l^2)}{v^4}+\frac{(-3+l^3)}{v^5}\right]}. \tag{8}$$

Theorem 3.2. The solution of the VIEs of the second kind $y(t)-\int_0^t y(\mathcal{T})sin(t-\mathcal{T})d\mathcal{T}=e^{-\frac{(t-l)^2}{2}}$ is expressed as

$$y(t)=\frac{e^{-\frac{l^2}{2}}}{120}[120+120lt+60l^2t^2+20(-2l+l^3)t^3+5(-l+l^2)t^4+(-3l+l^2)t^5]. \tag{9}$$

Proof: By taking ATM in equation (9) we get

$$\mathcal{A}[y(t)] - \mathcal{A}[\int_0^t y(\mathcal{T}) sin(t-\mathcal{T})\, d\,\mathcal{T}] = \mathcal{A}\left[e^{-\frac{(t-l)^2}{2}}\right]. \quad (10)$$

Using equation (2) the above equation becomes

$$\mathcal{A}[y(t)] - \mathcal{A}[y(t) * sint] = \mathcal{A}\left[e^{-\frac{(t-l)^2}{2}}\right]. \quad (11)$$

Also, by substituting convolution theorem and Theorem 3.1 in equation (11) we conclude

$$\mathcal{A}[y(t)]\left[1 - \frac{v}{v(v^2+1)}\right] = e^{\frac{-l^2}{2}\left[\frac{1}{v^2}+\frac{l}{v^3}+\frac{(-1+l^2)}{v^4}+\frac{(-3+l^3)}{v^5}\right]} \quad (12)$$

or

$$\mathcal{A}[y(t)] = e^{\frac{-l^2}{2}\left[\frac{1}{v^2}+\frac{l}{v^3}+\frac{(-1+l^2)}{v^4}+\frac{(-3+l^3)}{v^5}\right]} \times \left(\frac{v^2+1}{v^2}\right). \quad (13)$$

Based on equation (13), the partial fractional method can be used as follows:

$$\mathcal{A}[y(t)] = e^{\frac{-l^2}{2}\left[\frac{1}{v^2}+\frac{l}{v^3}+\frac{l^2}{v^4}+\frac{(-2l+l^3)}{v^5}+\frac{(-1+l^2)}{v^6}+\frac{(-3l+l^3)}{v^7}\right]}. \quad (14)$$

Then, Aboodh inverse transform can be used in equation (14) to get

$$y(t) = \frac{e^{\frac{-l^2}{2}}}{120}[120 + 120lt + 60l^2t^2 + 20(-2l + l^3)t^3 + 5(-l + l^2)t^4 + (-3l + l^2)t^5]. \quad (15)$$

In Theorem 3.2, we put h = 0.1 and l = 1 in the improved Simpson method. The analytical solution in equation (15) is compared with the numerical solution given in the table below. Computations were accomplished by using MATLAB.

Table1: Results for Analytical and Numerical Solutions

T	Numerical Solution	Analytical Solution
0.0	0.60653065971234	0.60653065972123
0.1	0.67011518945075	0.67011518946775
0.2	0.73915546247246	0.73915546247393
0.3	0.81302978485305	0.81302978485333

T	Numerical Solution	Analytical Solution
0.4	0.89108220143895	0.89108220143901
0.5	0.97266036523732	0.97266036523721
0.6	1.05700340680325	1.05700340681143
0.7	1.14332980362796	1.14332980362705
0.8	1.23077524951917	1.23077524951945
0.9	1.31839052399434	1.31839052399545
1.0	1.40512936166779	1.40512936167709

3.1 Comparison of Numerical and Analytical Solutions

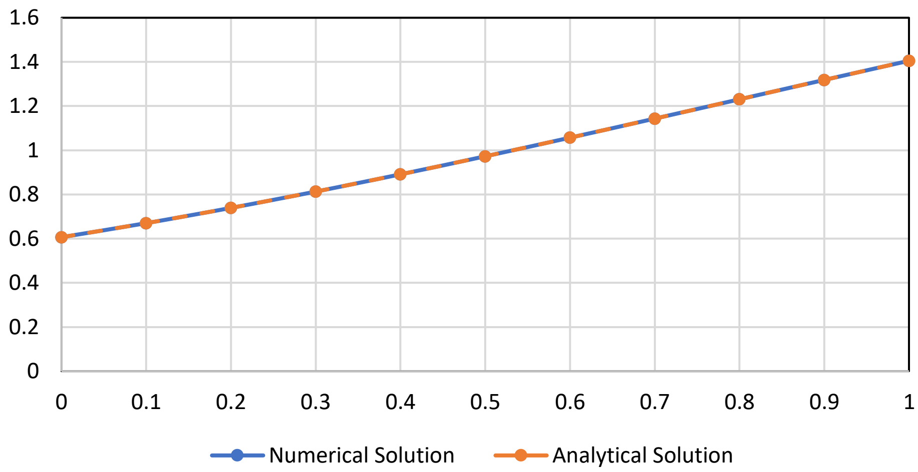

Figure 1. Numerical and analytical solutions of the example for h = 0.1 and l=1

4.CONCLUSION

This research is pertinent to the study of the second kind VIEs with bulge function. In this paper, analytical solution is obtained by using Aboodh transform, Aboodh inverse transform, convolution theorem, and Taylor series expansion. For the numerical solution, improved Simpson approximation method is used. Both the methods used are compared and the results are presented in the form of a table and graph. It is concluded that the analytical solution is the better solution.

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