Scientific Inquiry and Review (SIR)
Volume 6 Issue 1, Spring 2022
ISSN $_{(P)}$: 2521-2427, ISSN $_{(E)}$: 2521-2435
Homepage: https://journals.umt.edu.pk/index.php/SIR

Article QR

Title:	**General Construction of Efficient Circular Weakly Balanced Repeated Measurements Designs**
Author (s):	Qaisar Mehmood[1], Kashif Rasheed[2], Khadija Noreen[2], Rashid Ahmed[2], Berihan R. Elemary[3]
Affiliation (s):	[1]Government Graduate College Bahawal Nagar, Pakistan [2]The Islamia University of Bahawalpur, Pakistan [3]Damietta University and NUB-Nahda University, Beni Suef, Egypt
DOI:	https://doi.org/10.32350/sir.61.05
History:	Received: January 06, 2022, Last Revised: March 04, 2022, Accepted: March 25, 2022, Published Online:
Citation:	Mehmood Q, Rasheed K, Noreen K, Ahmed R, Elemary BR. General construction of efficient circular weakly balanced repeated measurements designs. *Sci Inquiry Rev.* 2022;6(1):66-80. https://doi.org/10.32350/sir.61.05
Copyright:	© The Authors
Licensing:	This article is open access and is distributed under the terms of Creative Commons Attribution 4.0 International License
Conflict of Interest:	Author(s) declared no conflict of interest

A publication of
School of Science
University of Management and Technology, Lahore, Pakistan

General Construction of Efficient Circular Weakly Balanced Repeated Measurements Designs

Qaisar Mehmood[*1], Kashif Rasheed[2], Khadija Noreen[2], Rashid Ahmed[2] and Berihan R. Elemary[3]

[1]Department of Statistics, Government Graduate College Bahawal Nagar, Pakistan
[2]Department of Statistics, The Islamia University of Bahawalpur, Pakistan
[3]Department of Applied, Mathematical and Actuarial Statistics, Damietta University and NUB-Nahda University, Beni Suef, Egypt

[*]qaisarm11@gmail.com

Abstract

Weakly balanced repeated measurements designs (RMDs) are used to balance out residual effects where minimal balanced RMDs cannot be obtained. RMDs are equally important in periods of both equal and unequal sizes. There should be a general procedure to construct these designs. In this article, some generators are developed for the general construction of efficient minimal circular weakly balanced RMDs.

Keywords: carry over effects, direct effects, efficiency of residual effects, repeated measurement designs (RMDs)

Introduction

Repeated measurements designs (RMDs) have application in many branches of scientific inquiry, therefore, they should be chosen in a way that treatments can be compared efficiently in the presence of residual effects. Minimal balanced RMDs (BRMDs) are useful to balance these effects economically. Minimal weakly balanced RMDs (MWBRMDs) are used in situations where minimal BRMDs cannot be obtained. An RMD is minimal weakly balanced if each treatment follows once and no time, or once and twice by every other treatment (excluding itself). Williams [1] introduced RMDs. Cheng and Wu [2] and Magda [3] constructed circular balanced RMDs (CBRMDs). Afsarinejad [4] and Iqbal and Jones [5] constructed BRMDs and SBRMDs with unequal period sizes. Sharma et al. [6] gave/developed a general construction of BRMDs for v odd. Iqbal and Tahir [7] presented circular SBRMDs (CSBRMD). Iqbal et al. [8] constructed some first- and second-order CBRMDs and CSBRMDs. Bashir et al. [9] and Rajab et al. [10] developed a series of CBRMDs in equal period sizes.

Rasheed et al. [11] and Daniyal et al. [12] developed a series of minimal CSBRMDs.

Minimal CWBRMDs (MCWBRMDs) were constructed by (i) [13] for $p = v$, (ii) [14] for $p \leq v$, (iii) [15] for p_1, p_2, and (iv) [16] for unequal period sizes. MCWBRMDs are equally important in periods of equal and unequal sizes. There should be a general procedure to construct these designs in periods of both equal and unequal sizes. In this article, some generators are developed for general construction of MCWBRMDs in periods of k different sizes. Here, MCWBRMD in which $v/2$ ordered pairs (i) do not appear are named as MCWBRMDs-I and (ii) appear twice are named as MCWBRMDs-II. An important feacture of this general construction is that the experimenter may obtain these designs in periods of (i) equal sizes by putting k = 1, (ii) two different sizes by putting k = 2, and so on. All these designs possess a high efficiency of separability as well as that of residual effects.

2. Efficiency of CWBRMDs

In this section, the efficiency of separability and the residual effects are discussed for CWBRMDs.

2.1 Efficiency for Residual Effects (Er)

Harmonic mean of non-zero eigen values of their respective information matrix is known as the efficiency for ER [17].

2.2 Efficiency of Separability (Es)

Divecha and Gondaliya's [18] formula to determine the efficiency of seprability (Es) is modified here for WBRMDs. Its modified form is given below.

$$ES = \left[1 - 1/(v\sqrt{v-1})\right] \times 100\%.$$

3. Method of Cyclic Shifts

This method was introduced by Iqbal [19]. Its Rule I is explained here for MCWBRMDs

Let $S_1 = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$ and $S_2 = [q_{i1}, q_{i2}, \ldots, q_{i(s-1)}]$.

- If $1 \leq q_{ij} \leq v-1$, and each of 1, 2, …, v-1 appears once in S^*, design is MCBRMD, otherwise MCWBRMD.

- If $0 \leq q_{ij} \leq v$-1, and each of 0, 1, …, v-1 appears once in S^*, the design is MCSBRMD, where S^* contains (i) all elements of all sets and (ii) [v-(sum of a set) mod v] for all sets.

Example 3.1: S_1 = [1,3,2,9,8] and S_2 = [4,5,6] produce MCWBRMD-II for v = 10 in p_1 = 6 and p_2 = 4.

Since S_1* = [1,3,2,9,8,7] and S_2* = [4,5,6,5], therefore, S* = [1,3,2,9,8,7,4,5,6,5], each of 1, 2,…, 9 appears once except 5 which appears twice. So, it is MCWBRMD-II, which is obtained cyclically as follows:

Take v subjects for S1 = [1,3,2,9,8]. Consider 0, 1,…, v-1 as elements of Period I, respectively. For elements of Period II, add 1 (mod 10) to each element of Period I. For elements of Period III, add 3 (mod 10) to each element of Period II. Similarly, add 2, 9, and 8, as shown in Table 1.

Table 1. Array from S1 = [1,3,2,9,8]

p	**Subjects**									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	1	2	3	4	5	6	7	8	9
2	1	2	3	4	5	6	7	8	9	0
3	4	5	6	7	8	9	0	1	2	3
4	6	7	8	9	0	1	2	3	4	5
5	5	6	7	8	9	0	1	2	3	4
6	3	4	5	6	7	8	9	0	1	2

Take v more experimental subjects for S2 = [4,5,6] and complete the array, see Table 2.

Table 2. Array from S2 = [4,5,6]

P	**Subjects**									
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	0	1	2	3	4	5	6	7	8	9
2	4	5	6	7	8	9	0	1	2	3

P	Subjects									
	11	12	13	14	15	16	17	18	19	20
3	9	0	1	2	3	4	5	6	7	8
4	5	6	7	8	9	0	1	2	3	4

Tables 1 and 2 together produce MCWBRMD for $v = 10$, $p_1 = 6$, and $p_2 = 4$.

4. Generators for MCWBRMDs-I

Generator 4.1: If $v = 2m_1i+2m_2+\ldots+2m_k+2$, $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3,\ldots, p_k = 2m_k$, then MCWBRMDs-I can be obtained from the following i sets for p_1 and one set each for all other ps using Rule I.

$S_{j+1} = [m_1j+1, m_1j+2, \ldots , m_1j+m_1, v\text{-}(m_1j+1), v\text{-}(m_1j+2), \ldots, v\text{-}(m_1j+m_1\text{-}1)]$;

$S_{1+i} = [m_1i+1, m_1i+2, \ldots , m_1i+m_2, v\text{-}(m_1i+1), v\text{-}(m_1i+2), \ldots, v\text{-}(m_1i+m_2\text{-}1)]$

$S_{2+i} = [m_1i+m_2+1, m_1i+m_2+2, \ldots , m_1i+m_2+m_3, v\text{-}(m_1i+m_2+1),$
$v\text{-}(m_1i+m_2+2), , v\text{-}(m_1i+m_2+m_3\text{-}1)]$

$S_{i+k-1} = [m_1i+m_2+\ldots+m_{k-1}+1, m_1i+m_2+\ldots+m_{k-1}+2,\ldots, m_1i+m_2+\ldots+m_k, v\text{-}(m_1i+m_2+\ldots+m_{k-1}+1), v\text{-}(m_1i+m_2+\ldots+m_{k-1}+2), ,$
$v\text{-}(m_1i+m_2+\ldots+m_k\text{-}1)]$

Corollary 4.1.1.

Putting $m_2 = m_3 = \ldots = m_k = 0$, using only S_{j+1}, Generator 4.1 produces MCWBRMDs-I in $p_1 = p = 2m_1$. Examples are given below in Table 3.

Table 3. MCWBRMDs-I Obtained from Generator 4.1

v	p	Sets	Es	Er
10	4	[1,2,9]+[3,4,7]	0.84	0.81
14	6	[1,3,2,13,12]+[4,5,6,10,9]	0.89	0.89
20	6	[1,3,2,19,18]+[4,5,6,16,15]+[7,8,9,13,12]	0.93	0.87
18	8	[1,2,3,4,17,16,15]+[5,6,8,13,7,12,11]	0.92	0.92

Corollary 4.1.2. Putting $m_3 = m_4 = \ldots = m_k = 0$, using only S_{j+1} and S_{i+1}, Generator 4.1 produces MCWBRMDs-I in $p_1 = 2m_1$ and $p_2 = 2m_2$. Examples are given below in Table 4.

Table 4. MCWBRMDs-I Obtained from Generator 4.1

v	p_1	p_2	Sets	Es	Er
12	6	4	[1,3,2,11,10]+[4,5,8]	0.87	0.87
18	6	4	[1,3,2,17,16]+[4,5,6,14,13]+[7,8,11]	0.92	0.86
14	8	4	[1,2,3,4,13,12,11]+[5,6,9]	0.89	0.89
16	8	6	[1,2,3,4,15,14,13]+[5,6,7,11,10]	0.91	0.91

Corollary 4.1.3. Putting $m_4 = m_5 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1} and S_{i+2}, Generator 4.1 produces MCWBRMDs-I in $p_1 = 2m_1$, $p_2 = 2m_2$ and $p_3 = 2m_3$. Examples are given below in Table 5.

Table 5. MCWBRMDs-I Obtained from Generator 4.1

v	p_1	p_2	p_3	Sets	Es	Er
20	8	6	4	[1,2,3,4,19,18,17]+[5,6,7,15,14]+[8,9,12]	0.93	0.87
28	8	6	4	[1,2,3,4,27,26,25]+[5,6,7,8,23,22,21]+[9,10,11,19,18]+ [12,13,16]	0.95	0.87
22	10	6	4	[1,2,3,4,5,21,20,19,18]+[6,7,8,16,15]+[9,10,13]	0.93	0.88
26	10	8	6	[1,2,3,4,5,25,24,23,22]+[6,7,8,9,20,19,18]+[10,11,12,16,15]	0.94	0.91

Corollary 4.1.4. Putting $m_5 = m_6 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2} and S_{i+3}, Generator 4.1 produces MCWBRMDs-I in $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3$ and $p_4 = 2m_4$. Similarly, for k = 5, 6, and so on. Examples are given below in Table 6.

Table 6. MCWBRMDs-I Obtained from Generator 4.1

v	p_1	p_2	p_3	p_4	Sets	Es	Er
30	10	8	6	4	[1,2,3,4,5,29,28,27,26]+[6,7,8,24,9,23, 22]+[10,11,12,20,19]+[13,14,17]	0.95	0.88
40	10	8	6	4	[1,2,3,4,5,39,38,37,36]+[6,7,8,9,34,10, 33,32,31]+[11,12,13,14,29,28,27]+ [15,16,17,25,24]+[18,19,22]	0.96	0.89
32	12	8	6	4	[1,2,4,3,5,6,31,30,28,29,27]+[7,8,9,10, 25,24,23]+[11,12,13,21,20]+[14,15,18]	0.95	0.89
38	12	10	8	6	[1,2,4,3,6,5,37,36,35,34,33]+[8,7,9,10, 11,30,31,29,28]+[12,13,14,15,26,25,24]+ [16,17,18,22,21]	0.95	0.88

Generator 4.2: If $v = 2m_1i+4m_2+2m_3+\ldots+2m_k+2$, $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3$, …, $p_k = 2m_k$, then MCWBRMDs-I can be obtained with two sets for p_2 using Rule I.

$S_{j+1} = [m_1j+1, m_1j+2,\ldots, m_1j+m_1, v-(m_1j+1), v-(m_1j+2), \ldots, v-(m_1j+m_1-1)]$;

$S_{1+i} = [m_1i+1, m_1i+2,\ldots, m_1i+m_2, v-(m_1i+1), v-(m_1i+2), \ldots, v-(m_1i+m_2-1)]$

$S_{2+i} = [m_1i+m_2+1, m_1i+m_2+2,\ldots, m_1i+m_2+m_3, v-(m_1i+m_2+1), v-(m_1i+m_2+2), \ldots, v-(m_1i+m_2+m_3-1)]$

$S_{3+i} = [m_1i+2m_2+1, m_1i+2m_2+2,\ldots, m_1i+2m_2+m_3, v-(m_1i+2m_2+1),$

$v-(m_1i+2m_2+2), \ldots,v-(m_1i+2m_2+m_3-1)$

$S_{i+k}=[m_1i+2m_2+m_3+\ldots+m_{k-1}+1,m_1i+2m_2+m_3+\ldots$

$+ m_{k-1}+2,\ldots, m_1i+2m_2+m_3+\ldots+m_k,$

$v-(m_1i+2m_2+m_3+\ldots+m_{k-1}+1), v-(m_1i+2m_2+m_3+\ldots+m_{k-1}+2), \ldots,$

$v-(m_1i+2m_2+m_3+\ldots+m_{k-1} -1)]$

Corollary 4.2.1. Putting $m_3 = m_4 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1} and Examples are given below in Table 7.

Table 7. MCWBRMDs-I Obtained from Generator 4.2

v	p_1	p_2	Sets	Es	Er
16	6	4	[1,3,2,15,14]+[4,5,12]+[6,7,10]	0.91	0.83
18	8	4	[1,2,3,4,17,16,15]+[5,6,13]+[7,8,11]	0.92	0.86
22	8	6	[1,2,3,4,21,20,19]+[5,6,7,17,16]+[8,9,10,14,13]	0.93	0.89
28	10	8	[1,2,3,4,5,27,26,25,24]+[6,7,8,9,22,21,20]+ [10,11,12,13,18,17,16]	0.95	0.91

Corollary 4.2.2. Putting $m_4 = m_5 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2} and S_{i+3}, Generator 4.2 produces MCWBRMDs-I in $p_1 = 2m_1$, $p_2 = 2m_2$ and $p_3 = 2m_3$. Examples are given below in Table 8.

Table 8. MCWBRMDs-I Obtained from Generator 4.2

v	p_1	p_2	p_3	Sets	Es	Er
26	8	6	4	[1,2,3,4,25,24,23]+[5,6,7,21,20]+ [8,9,10,18,17]+[11,12,15]	0.94	0.86
28	10	6	4	[1,2,3,4,5,27,26,25,24]+[6,7,8,22,21]+ [9,10,11,19,18]+[12,13,16]	0.95	0.87
32	10	8	4	[1,2,3,4,5,31,30,29,28]+[6,7,8,9,26,25,24]+ [10,11,12,13,22,21,20]+[14,15,18]	0.95	0.97
34	10	8	6	[1,2,3,4,5,33,32,31,30]+[6,7,8,9,28,27,26]+ [10,11,12,13,24,23,22]+[14,15,16,20,19]	0.96	0.90

Corollary 4.2.3. Putting $m_5 = m_6 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2}, S_{i+3} and S_{i+4}, Generator 4.2 produces MCSPBRMDs-I in $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3$, and $p_4 = 2m_4$. Similarly, for k = 5, 6, and so on. Examples are given below in Table 9.

Table 9. MCWBRMDs-I Obtained from Generator 4.2

v	p_1	p_2	p_3	p_4	Sets	Es	Er
38	10	8	6	4	[1,2,3,4,5,37,36,35,34]+[6,7,8,9,32, 31,30]+[10,11,12,13,28,27,26]+ [14,15,16,24,23]+[17,18,21]	0.96	0.88
40	12	8	6	4	[1,2,4,3,6,5,39,38,37,36,35]+[7,8,9, 10,33,32,31]+[11,12,13,14,29,28,27]+ [15,16,17,25,24]+[18,19,22]	0.96	0.89
44	12	10	6	4	[1,2,4,3,6,5,43,42,41,40,39]+[7,8,9, 10,11,37,36,35,34]+[12,13,14,15,16, 32,31,30,29]+[17,18,19,27,26]+ [20,21,24]	0.97	0.90
48	12	10	8	6	[1,2,4,3,6,5,47,46,45,44,43]+[7,8, 9,10,11,41,40,39,38]+[12,13,14,15, 16,36,35,34,33]+[17,18,19,31,30, 28,29]+[21,20,22,27,26]	0.95	0.91

5. Generators for MCWBRMDs-II

Generator 5.1: If $v = 2m_1i+2m_2+\ldots+2m_k$, $p_1 = 2m_1$, $p_2 = 2m_2$, …, $p_k = 2m_k$, then MCWBRMDs-II can be constructed from the following i sets for p_1 and one set each for all other ps using Rule I.

$S_{j+1} = [m_1j+1, m_1j+2,\ldots, m_1j+m_1, v-(m_1j+1), v-(m_1j+2), \ldots, v-(m_1j+m_1-1)]$

$S_{1+i} = [m_1i+1, m_1i+2,\ldots, m_1i+m_2, v-(m_1i+1), v-(m_1i+2), \ldots, v-(m_1i+m_2-1)]$

$S_{2+i} = [m_1i+m_2+1, m_1i+m_2+2,\ldots, m_1i+m_2+m_3, v-(m_1i+m_2+1), v-(m_1i+m_2+2), \ldots, v-(m_1i+m_2+m_3-1)]$ …

$S_{i+k-1} = [m_1i+m_2+\ldots+m_{k-1}+1, m_1i+m_2+\ldots+m_{k-1}+2,\ldots, m_1i+m_2+\ldots+m_k, v-(m_1i+m_2+\ldots+m_{k-1}+1), v-(m_1i+m_2+\ldots+m_{k-1}+2), \ldots,v-(m_1i+m_2+\ldots+m_k -1)]$

Corollary 5.1.1. Putting $m_2 = m_3 = \ldots = m_k = 0$, using only S_{j+1}, Generator 5.1 produces MCWBRMDs-II in $p_1 = p = 2m_1$. Examples are given below in Table 10.

Table 10. MCWBRMDs-II obtained from Generator 5.1 in periods of equal sizes

v	*p*	Sets	Es	Er
8	4	[1,2,7]+[3,4,5]	0.82	0.84
12	6	[1,3,2,11,10]+[4,5,6,8,7]	0.88	0.91
18	6	[1,3,2,17,16]+[4,5,6,14,13]+[7,8,9,11,10]	0.92	0.88
16	8	[1,2,3,4,15,14,13]+[6,5,7,8,11,10,9]	0.91	0.93

Corollary 5.1.2. Putting $m_3 = m_4 = \ldots = m_k = 0$, using only S_{j+1} and S_{i+1}, Generator 5.1 produces MCWBRMDs-II in $p_1 = 2m_1$ and $p_2 = 2m_2$. Examples are given below in Table 11.

Table 11. MCWBRMDs-II Obtained from Generator 5.1

v	p_1	p_2	Sets	Es	Er
10	6	4	[1,3,2,9,8]+[4,5,6]	0.86	0.87
12	8	4	[1,2,3,4,11,10,9]+[5,6,7]	0.88	0.89
14	8	6	[1,2,3,4,13,12,11]+[5,6,7,9,8]	0.90	0.92
22	8	6	[1,2,3,4,21,20,19]+[5,6,7,8,17,16,15]+ [9,10,11,13,12]	0.94	0.90

Corollary 5.1.3. Putting $m_4 = m_5 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1} and S_{i+2}, Generator 5.1 produces MCWBRMDs-II in $p_1 = 2m_1$, $p_2 = 2m_2$, and $p_3 = 2m_3$. Examples are given below in Table 12.

Table 12. MCWBRMDs-II Obtained from Generator 5.1

v	p_1	p_2	p_3	Sets	Es	Er
18	8	6	4	[1,2,3,4,17,16,15]+[5,6,13,7,12]+ [8,9,10]	0.92	0.88
20	10	6	4	[1,2,3,4,5,19,18,17,16]+[6,7,8,14,13]+ [9,10,11]	0.93	0.89
22	10	8	4	[1,2,3,4,5,21,20,19,18]+[6,7,8,9,16,15, 14]+[10,11,12]	0.94	0.90
24	10	8	6	[1,2,3,4,5,23,22,21,20]+[7,6,8,9,17,18, 16]+[10,11,12,14,13]	0.94	0.91

Corollary 5.1.4. Putting $m_5 = m_6 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2} and S_{i+3}, Generator 5.1 produces MCWBRMDs-II in $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3$, and $p_4 = 2m_4$. Similarly, for k = 5, 6, and so on. Examples are given below in Table 13.

Table 13. MCWBRMDs-II Obtained from Generator 5.1

v	p_1	p_2	p_3	p_4	Sets	Es	Er
28	10	8	6	4	[1,2,3,4,5,27,26,25,24]+[6,7,8,9,22,21,20]+[10,11,12,18,17]+[13,14,15]	0.95	0.92
38	10	8	6	4	[1,2,3,4,5,37,36,35,34]+[6,7,8,9,10,32,31,30,29]+[11,12,13,14,27,26,25] +[15,16,17,23,22]+[18,19,20]	0.96	0.91
30	12	8	6	4	[1,2,4,3,6,5,29,28,27,26,25]+[7,8,9,10,23,22,21]+[11,12,13,19,18]+[14,15,16]	0.95	0.89
36	12	10	8	6	[1,2,4,3,6,5,35,34,33,32,31]+[7,8,9,10,11,29,28,27,26]+[12,13,14,15,24,23,22]+[16,17,18,20,19]	0.96	0.93

Generator 5.2: If $v = 2m_1i+4m_2+2m_3+\ldots+2m_k$, $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3,\ldots, p_k = 2m_k$, then MCWBRMDs-II can be constructed with two sets for p_2 using Rule I.

$S_{j+1} = [m_1j+1, m_1j+2,\ldots, m_1j+m_1, v-(m_1j+1), v-(m_1j+2), \ldots, v-(m_1j+m_1-1)]$

$S_{1+i} = [m_1i+1, m_1i+2,\ldots, m_1i+m_2, v-(m_1i+1), v-(m_1i+2), \ldots, v-(m_1i+m_2-1)]$

$S_{2+i} = [m_1i+m_2+1, m_1i+m_2+2,\ldots, m_1i+m_2+m_3, v-(m_1i+m_2+1), v-(m_1i+m_2+2), \ldots, v-(m_1i+m_2+m_3-1)]$

$S_{3+i} = [m_1i+2m_2+1, m_1i+2m_2+2,\ldots, m_1i+2m_2+m_3, v-(m_1i+2m_2+1), v-(m_1i+2m_2+2), \ldots,$

$v-(m_1i+2m_2+m_3-1)]$ …

$S_{i+k} = [m_1i+2m_2+m_3+\ldots+m_{k-1}+1, m_1i+2m_2+m_3+\ldots+m_{k-1}+2,\ldots, m_1i+2m_2+m_3+\ldots+m_k,$

$v-(m_1i+2m_2+m_3+\ldots+m_{k-1}+1), v-(m_1i+2m_2+m_3+\ldots+m_{k-1}+2), \ldots,v-(m_1i+2m_2+m_3+\ldots+m_{k-1} -1)]$

Corollary 5.2.1. Putting $m_3 = m_4 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1} and S_{i+2}, Generator 5.2 produces MCWBRMDs-II in $p_1 = 2m_1$ and $p_2 = 2m_2$. Examples are given below in Table 14.

Table 14. MCWBRMDs-II Obtained from Generator 5.2

v	p_1	p_2	Sets	Es	Er
14	6	4	[1,3,2,13,12]+[4,5,10]+[6,7,8]	0.90	0.84
16	8	4	[1,2,3,4,15,14,13]+[5,6,11]+[7,8,9]	0.91	0.86
20	8	6	[1,2,3,4,19,18,17]+[5,6,7,15,14]+[8,9,10,12, 11]	0.93	0.89
22	10	6	[1,2,3,4,5,21,20,19,18]+[6,7,8,16,15]+[9,10, 11,13,12]	0.94	0.90

Corollary 5.2.2. Putting $m_4 = m_5 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2} and S_{i+3}, Generator 5.2 produces MCWBRMDs-II in $p_1 = 2m_1$, $p_2 = 2m_2$ and $p_3 = 2m_3$. Examples are given below in Table 15.

Table 15. MCWBRMDs-II Obtained from Generator 5.2

v	p_1	p_2	p_3	Sets	Es	Er
24	8	6	4	[1,2,3,4,23,22,21]+[5,6,7,19,18]+[8,9,10, 16,15]+[11,12,13]	0.94	0.87

v	p_1	p_2	p_3	Sets	Es	Er
26	10	6	4	[1,2,3,4,5,25,24,23,22]+[6,7,8,20,19]+ [9,10,11,17,16]+[12,13,14]	0.95	0.88
30	10	8	4	[1,2,3,4,5,29,28,27,26]+[6,7,8,24,9, 23,22]+[10,11,12,13,20,19,18]+[14,15,16]	0.95	0.89
32	10	8	6	[1,2,3,4,5,31,30,29,28]+[6,7,8,9,26,25,24] +[10,11,12,13,22,21,20]+[14,15,16,18,17]	0.96	0.90

Corollary 5.2.3. Putting $m_5 = m_6 = \ldots = m_k = 0$, using only S_{j+1}, S_{i+1}, S_{i+2}, S_{i+3} and S_{i+4}, Generator 5.2 produces MCWBRMDs-II in $p_1 = 2m_1$, $p_2 = 2m_2$, $p_3 = 2m_3$, and $p_4 = 2m_4$. Similarly, for k = 5, 6, and so on. Examples are given below in Table 16.

Table 16. MCWBRMDs-II Obtained from Generator 5.2

v	p_1	p_2	p_3	p_4	Sets	Es	Er
36	10	8	6	4	[1,2,3,4,5,35,34,33,32]+[6,7,8,9,30, 29,28]+[10,11,12,13,26,25,24]+[14,15, 16,22,21]+[17,18,19]	0.96	0.90
38	12	8	6	4	[1,2,4,3,6,5,37,36,35,34,33]+[7,8,9, 10,31,30,29]+[11,12,13,14,27,26,25]+ [15,16,17,23,22]+[18,19,20]	0.96	0.90
42	12	10	6	4	[1,2,4,3,6,5,41,40,39,38,37]+[7,8,9,10, 11,35,34,33,32]+[13,12,14,15,16,29,30, 28,27]+[17,18,19,25,24]+[20,21,22]	0.97	0.90
46	12	10	8	6	[1,2,4,3,6,5,45,44,43,42,41]+[7,8,9,10, 11,39,38,37,36]+[13,12,14,15,16,34, 33,32,31]+[17,18,19,29,20,28,27]+ [21,22,23,25,24]	0.97	0.91

6. Conclusion

Generators have been developed in this study for the easy construction of MCWBRMDs in periods of unequal sizes. The experimenter may obtain these designs in (i) equal period sizes by putting k = 1, (ii) two different period sizes by putting k = 2, and so on. All these designs possess a high efficiency of separability as well as that of residual effects. Therefore, our proposed designs are efficient enough (i) to balance the residual effects and (ii) to estimate the direct effects and residual effects, independently.

Acknowledgement:

Athors are thankful to the anonymous reviewers for their valuable corrections and suggestions.

References

[1] Williams ER. Experimental designs balanced for the estimation of residual effects of treatments. *Aus J Sci Res*. 1949;2(2):149-168. https://doi.org/10.1071/CH9490149
[2] Cheng CS, Wu CF. Balanced repeated measurements designs. *Ann Stat*. 1980;8(6):1272-1283. https://doi.org/10.1214/aos/1176345200
[3] Magda CG. Circular balanced repeated measurements designs. *Commun Stats Theory Methods*. 1980;9(18):1901-1918. https://doi.org/10.1080/03610928008828004
[4] Afsarinejad K. Repeated measurements designs with unequal periods sizes. *J Italian Stat Soc*. 1994;2:161-168. https://doi.org/10.1007/BF02589224
[5] Iqbal I, Jones B. Efficient repeated measurements designs with equal and unequal period sizes. *J Stat Plan Infer*. 1994;42(1-2):79-88. https://doi.org/10.1016/0378-3758(94)90190-2
[6] Sharma VK, Jaggi S, Varghese C. Minimal balanced repeated measurements designs. *J Appl Stat*. 2003;30(8):867-872. https://doi.org/10.1080/0266476032000075958
[7] Iqbal I, Tahir MH. Circular strongly balanced repeated measurements designs. *Commun Stats Theory Methods*. 2009;38(20):3686-3696. https://doi.org/10.1080/03610920802642566
[8] Iqbal I, Tahir MH, Ghazali SSA. Circular first-and second-order balanced repeated measurements designs. *Commun Stats Theory Methods*. 2010;39(2):228-240. https://doi.org/10.1080/03610920902941728
[9] Bashir Z, Ahmed R, Tahir MH, Ghazali SSA, Shehzad F. Some extensions of circular balanced and circular strongly balanced repeated measurements designs. *Commun Stats Theory Methods*. 2018;47(9):2183-2194.https://doi.org/10.1080/03610926.2017.133140

[10] Rajab MR, Ahmed R, Shehzad F, Tahir MH. Some new constructions of circular balanced repeated measurements designs. *Commun Stats Theory Methods*. 2018;47(17):4142-4151. https://doi.org/10.1080/03610926.2017.1367817
[11] Rasheed U, Rasheed HMK, Rasheed M, Ahmed R. Minimal circular strongly balanced repeated measurements designs in periods of three different sizes. *Commun Stats Theory Methods*. 2018;47(16):4088-4094. https://doi.org/10.1080/03610926.2017.1361995
[12] Daniyal M, Ahmed R, Shehzad F, Tahir MH, Iqbal Z. Construction of repeated measurements designs strongly balanced for residual effects. *Commun Stats Theory Methods*. 2020;49(17):4288-4297. https://doi.org/10.1080/03610926.2019.1599019
[13] Bailey RA, Cemeron PJ, Fillipiak K, Kunert J, Markiewicz A. On optimality and construction of circular repeated measurements designs. *Stat Sinica*. 2017;27(1):1-22. http://dx.doi.org/10.5705/ss.202015.005
[14] Khan A, Ahmed R, Shehzad F, Tahir MH, Ghazali SSA. Construction of circular partially balanced repeated measurement designs. *Commun Stat Simul Computa*. 2019;48(2):506-515. https://doi.org/10.1080/03610918.2017.1387661
[15] Hussain S, Ahmed R, Aslam M, Shah A, Rasheed HMK. Some new construction of circular weakly balanced repeated measurements designs in periods of two different sizes. *Commun Stats Theory Methods*, 2020;49(9):2253-2263. https://doi.org/10.1080/03610926.2019.1570263
[16] Hussain S, Gondaliya J, Ahmed R., Jabeen R. Some new generators to obtain efficient circular weakly balanced repeated measurements designs. *Commun Stats Theory Methods*. 2021;50(15):3713-3730. https://doi.org/10.1080/03610926.2019.1570263
[17] Pearce SC, Calinski T, Marshall TFdeC. The basic contrasts of an experimental design with special reference to the analysis of data. *Biometrika*. 1974;61(3):449-460. https://doi.org/10.1093/biomet/61.3.449
[18] Divecha J, Gondaliya J. Construction of minimal balanced crossover designs having good efficiency of separability. *Elec J Stat*. 2014;8:2923-2936. https://doi.org/10.1214/14-EJS979
[19] Iqbal I. *Construction of experimental designs using cyclic shifts*. Dessertation. University of Kent at Canterbury;1991. https://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.293152