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On the Existence of Fault-Tolerant Numbers of Two-Bridge Knots

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ABSTRACT

DNA is a hereditary material of every cell which is the fundamental molecule in the transfer of encrypted information about cells from parent cells to daughter cells. The long-chain DNA molecule is tightly tangled to fit in the nucleus. The supercoiled DNA resembles a complicated knot in the structure. Every cellular process involving DNA like recombination, catenation, de-catenation, replication, and sequencing is initiated by the relaxation of the supercoiled DNA using enzyme action. The enzyme activity, which is used to untangle DNA is analogous to the process of unknotting a knot. The unknotting number refers to the least number of enzyme topoisomerase II actions. A new variant of the unknotting number namely fault-tolerant unknotting number ensures the un-entanglement of DNA knot in spite of the failure of enzyme activity at some point. This article investigates, a special family of knots named two-bridge knots for the existence of fault-tolerant unknotting numbers. Furthermore, the current study explores the subfamilies of two-bridge knots for fault-tolerant unknotting numbers. Besides this, two special subfamilies, $C(a, a_1, a_2, a_3, \dots, \pm 2, -a_k, -a_{k-1}, \dots, -a_2, -a_1)$ *and* $C(2n, 3)$ *of two-bridge knots with unknotting number 1 and* n *respectively are explored for a variant of fault-tolerant unknotting number called restricted fault-tolerant unknotting number. Lastly, this paper concludes by explicating biological disposition of fault-tolerant unknotting numbers in terms of enzyme action on the DNA.*

***Keywords*:** enzyme activity, fault-tolerant numbers, knot, two-bridge knot, unknotting numbers

INTRODUCTION

The theory of knots is a combination of geometry and topology whose fundamental object is a mathematical knot. Knots are notable mathematical objects, present far and wide around us, beneficial to anchor boats and to tie our shoe laces. The core of knot theory is to distinguish like and unlike knots. To serve this purpose, a knot invariant is assigned to a knot that may be just a number like the crossing number, the unknotting number, the bridge number of a knot or a complicated polynomial like jones polynomial, Alexander polynomial, and m-polynomial (see [1, 2] for further detail).

The association of knot theory to molecular biology is initiated by the realization of a double helical structure of DNA. The revelation of the model of DNA as a rational knot [3] straightened out the road to the mathematical study of DNA. The long-chain double helix of DNA is regulated naturally by limiting it to specific proteins named histones that assist in DNA supercoiling. The model of supercoiled DNA over histones is baptized as beads-on-a-string. The tightly knotted DNA is packed into chromosomes that reside in the nucleus of a cell. DNA carries all the instructions necessary for an organism to survive, reproduce, and maintain its uniqueness. The key functions of DNA in the cell include:

- *Replication*: by which copies of DNA are generated in the process of cell division [4]
- *Transcription*: in which coded information on DNA is decoded [4]
- *Recombination*: in which DNA pieces are split and realigned to generate new combinations for genetic diversity [5]
- *Sequencing*: refers to some laboratory techniques for pattern recognition of bases in DNA [6]

The unwinding of tightly packed DNA is a prerequisite for above mentioned cellular functions of DNA. The enzyme topoisomerase type-I (type-II) serves the un-entanglement of DNA by cleaving one (two resp.) strands of double helix and permitting these strands to go through a cleavage point before rejoining [7]. This enzyme action that catalyzes every process involving DNA, can be referred to unknotting operation in the tangle model of DNA and the least number of such enzyme actions to unknot supercoiled DNA gives an unknotting number of DNA knots. During the phenomenon of DNA unwinding, there are chances of enzyme failure to perform an unknotting operation. The fault-tolerant unknotting number (see definition 2. 4) is a minimum number of enzyme failures that the system can bear and still unknot DNA. The fault-tolerant unknotting set refers to the group of enzymes that are able to unknot DNA even if an enzyme fails to perform its action at some site in DNA. In 1954 and 1956, Schubert [8, 9] put forward the notion of knots by assigning a rational number to each member of the family. By a continued fraction of that rational number, the canonical form or two-bridge form of the rational knot was procured. The connection between the tangle model of DNA as a rational knot and the two-bridge form of a rational knot attracted many researchers to explore this family of knots. Later in 1998, Torisu worked to probe those two-bridge knots whose Gordian distance is one [10]. Besides exploring the geometry of two-bridge knots, many authors like Hartley and Kanenobu worked on polynomial invariants of two-bridge knots [11, 12]. In 2004, Wit computed number of two-bridge knots up to 16 crossings [13] and an analogous result for two-bridge links were proved by Demir [14].

In this paper, the researcher tried to explore the family of two-bridge knots for a new invariant of knots, named fault-tolerant unknotting number, introduced in [15]. The authors pose the following conjectures in [15].

1. Every knot diagram has a fault-tolerant set (strong version).
2. Every knot has a diagram with a fault-tolerant set (weak version).

In [15], the authors computed and compared the unknotting number and the fault-tolerant unknotting number for several knots and knot families and concluded that unknotting number is always less than the proposed fault-tolerant unknotting number. In the same article the authors also proved that the fault-tolerant unknotting number is a knot invariant as it distinguishes between the like and unlike knots. For example, according to **Table 1** in [15], the trefoil knot 3_1 and the knot 7_7 having the same unknotting number one, have fault-tolerant unknotting numbers 2 and 3, respectively. As a result, fault-tolerant unknotting number serves as a strong knot invariant than unknotting number as it distinguishes knots having the same unknotting number. Brainstorming to solve the above said conjectures lead to introducing and proving the existence of two new variants, named the weak fault-tolerant number $wt(K)$ and the restricted fault-tolerant number $rt(K)$ in which the previous definition of the fault-tolerant unknotting number for a classical knot K was altered [16].

The remaining paper is organized as; section 2 comprises some introductory notions and the formation of rational knots from the tangle. In section 3, new results about fault-tolerance of some subfamilies of two-bridge knots are stated and proved. Section 4 provides an interpretation of the fault-tolerant numbers from a biological perspective and section 5 comprises of the conclusion.

2. ESSENTIAL CONCEPTS

A review of some precursory conceptions of knot theory is stated in this section from [1]. Let K represent a knot, D be the representation of a diagram of a knot, and $C(D)$ be the set of all crossings of D.

Definition 2.1. [1] An unknotting operation or crossing switch in D involves changing an over-crossing to an under-crossing or vice versa. For a knot diagram D, an unknotting set U is a subset of $C(D)$ such that by switching the crossings in U, D transforms into the un-knot.

Definition 2.2. [1] The cardinality of the smallest unknotting set is the unknotting number $u(D)$ of D. The unknoting number of a knot K is defined as $u(K) = \min\{|u(D)|: [D] = K\}$

Where [D] denotes the class of all diagrams representing K.

Definition 2.3. [17] A minimal knot diagram is one that cannot be drawn with fewer crossings.

Definition 2.4. [15] A fault-tolerant unknotting set W is an unknotting set for which the set $W \setminus \{w\}$ is still an unknotting set for each $w \in W$. The fault-tolerant unknotting number $t(D)$ of a knot diagram is the minimum cardinality of a fault-tolerant unknotting set.

Definition 2.5. [16] A set $W \subseteq C(D)$ is called a weak fault tolerant set if for each $w \in W, W \setminus \{w\}$ contains an unknotting set.

Definition 2.6. [16] The weak fault-tolerant number wt (D) of a diagram D and wt (K) for a knot K are defined in the following manner:

$$wt(D) = \min\{|W|: W \subseteq C(D), W \text{ is a weak fault -tolerant set }\}$$

And

$$wt(K) = \min \{wt(D): [D] = K\}$$

Definition 2.7. [16] A non-unknotting set $R \subseteq C(D)$ is called restricted fault-tolerant set if for each $r \in R, R \setminus \{r\}$ contains an unknotting set.

Definition 2.8. [16] The restricted fault-tolerant number $rt(D)$ for a diagram D and $rt(K)$ for a knot K is defined as follows:

$$rt(D) = \min\{|R|: R \subseteq C(D), R \text{ is a restricted fault-tolerant set}\}.$$

and

$rt(K) = \min\{rt(D): [D] = K\}$

Theorem 2.1: [16] If $u(K) = 1$ for a knot K such that D is a diagram of K with at least two unknotting sets of cardinality 1, then $\mathrm{wt(K)} = 2$. Otherwise, $\mathrm{wt(K)} = 3$

From now onward, both the weak fault-tolerant number and the restricted fault-tolerant number would be collectively termed as fault-tolerant number.

Remark: It is worth mentioning here that every fault-tolerant unknotting set is an unknotting set but the converse is not necessarily true. Unlike a fault-tolerant unknotting set, the fault-tolerant sets (both weak and restricted) are not necessarily unknotting. A weak fault-tolerant set may be unknotting, therefore, each fault-tolerant unknotting set is also a weak fault-tolerant set but a restricted fault-tolerant set must be strictly non-unknotting. Furthermore, it was observed that each restricted fault-tolerant set is also a weak fault-tolerant set [16].

2.1 Rational Knot and Conway Form of a Knot

The Conway notation for a knot is defined by J. H. Conway [18]. This notation for a knot K is $(c_1, c_2, c_3, \dots, c_n)$, composed of n components c_i, for all i, where c_i is a portion of knot termed as tangle. A tangle is a specific fragment of a knot that, when encircled in the projection plane, intersects the circle at four points necessarily. For visual representation of a tangle, assume two strings attached to the inside of a sphere that can slide around the boundary of the sphere. When strings are in horizontal position and do not cross each other, they are forming the zero tangle and strings in vertical position without crossing form the ∞-tangle. These strings can be twisted horizontally as well as vertically and a combination of alternating sequence of horizontal and vertical twists starting with horizontal twists, would tangle up to give a rational tangle that corresponds to a rational number. The rational number associated with a rational tangle comes from a continued fraction. The rule of writing a continued fraction is as follows:

- If the last twist is vertical then the continued fraction is:

$$0 + \cfrac{1}{last + \cfrac{1}{second\ last + \cfrac{1}{third\ last + \cdots + \cfrac{1}{first}}}}$$

- When the last twist is horizontal then the continued fraction is:

$$\cfrac{1}{last + \cfrac{1}{second\ last + \cfrac{1}{third\ last + \cdots + \cfrac{1}{first}}}}$$

For example, the tangle with two horizontal and three vertical twists is represented by (2, 3) and the resulting fraction is:

$$0 + \cfrac{1}{3 + \cfrac{1}{2}} = \frac{2}{7}$$

But if we have two horizontal, three vertical and four horizontal twists, the tangle is (2, 3, 4) and the resulting fraction is:

$$4 + \cfrac{1}{3 + \cfrac{1}{2}} = \frac{30}{7}$$

The numerator closure of a rational tangle $(c_1, c_2, c_3, \dots, c_n)$ ultimately yields a knot diagram in Conway notation as shown in **Figure 1** where $C(a_1, a_2, a_3, \dots, a_n)$ is a rational knot obtained by numerator closure of a tangle composed of $a_1, a_2, a_3, \ \dots, a_n$ twists. The Conway notation (or Conway diagram) of a knot has a minimal number of crossings, and it is unique among all diagrams of a knot (see [1] for further detail). Rational knots and rational links were first considered in [19, 20] and are a way of classifying knots and links by a rational number according to following theorem.

Theorem 2.2. [3] Two rational tangles are isotopic if they have the same fraction.

2.2 Two-Bridge Knot

In this section, two-bridge knots using Conway notation are defined as rational knots depicted in the **Figure 1** and **Figure 2**. In this form, any two-bridge knot or link is represented as $C(a_1, a_2, a_3, \ldots, a_n)$, where $a_1, a_2, a_3, \ldots, a_n$ are non-zero integers representing tangles whose numerator closure is the resulting two-bridge knot. In Conway form $C(a_1, a_2, a_3, \ldots, a_n)$, the continued fraction made up of $a_1, a_2, a_3, \ldots, a_n$ results a rational number $\frac{\alpha}{\beta}$ which is the slope of presentation $C(a_1, a_2, a_3, \ldots, a_n)$.

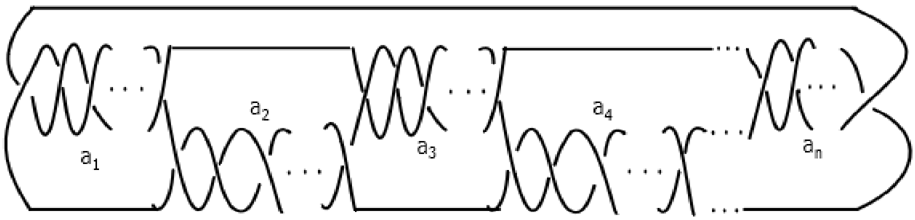

Figure 1. When n is odd

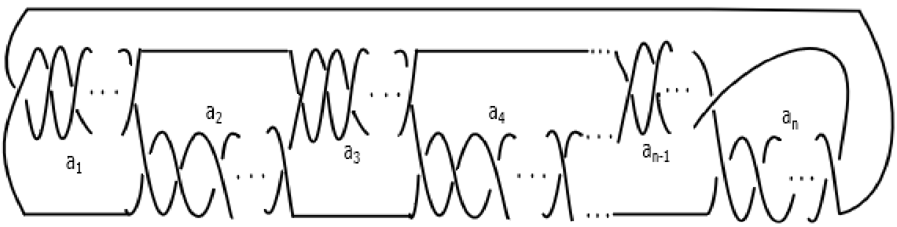

Figure 2. When n is even

The slope of a two-bridge knot is:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cdots + \cfrac{1}{a_n}}}}} = \frac{\alpha}{\beta}$$

When α is even, $C(a_1, a_2, a_3, \ldots, a_n)$ is a two-bridge link and for odd α, $C(a_1, a_2, a_3, \ldots, a_n)$ is a two-bridge knot. The instigation conducted to explore the family of two-bridge knots lead to the existence of fault-tolerant unknotting set for some subfamilies. Furthermore, the existence of restricted fault-tolerant unknotting number is asserted for two special subfamilies of two-bridge knots: the family $C(a, a_1, a_2, a_3, \ldots, a_k, \pm 2,$ -

$a_{k-1}, \ldots, -a_2, -a_1)$ with unknotting number 1 and the family $C(2n, 3)$ whose unknotting number is n [21].

3. NEW RESULTS

The family of two-bridge knots in Conway notation for fault-tolerant unknotting sets is explored in this section. Depending on values of n and $a_1, a_2, a_3, \ldots, a_n$, the researcher explored the following subfamilies of two-bridge knots.

3.1 The Family

$C(a_1, a_2, a_3, \ldots, a_n)$ with a_1 odd and $a_i's$ even for all $i \geq 2$ here, we take two-bridge knots of the form $C\ (2m_1 + 1, 2m_2, 2m_3, \ldots, 2m_n)$, for some integers $m_1, m_2, m_3, \ldots, m_n$. For n, we have the following cases:

Case 1: When n is odd

For odd n, we have a two-bridge knot of the form shown in **Figure 1**.

Theorem 3.1. The set V consisting of alternating crossings (see **Figure 3**) in $C(2m_1 + 1, 2m_2, 2m_3, \ldots, 2m_n\ )$ such that:

$$|V| = \frac{(2m_1 + 1) + 1}{2} + \frac{2m_2}{2} + \frac{2m_3}{2} + \cdots + \frac{2m_n}{2}$$

Is a fault-tolerant unknotting set of $(2m_1 + 1, 2m_2, 2m_3, \ldots, 2m_n\ )$.

Proof: Firstly, it is essential to prove that every subset of the set V with cardinality $|V|$-1 is an unknotting set. For this there are following possibilities:

- For some c in the tangle $2m_1 + 1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 5**(**a**) which is unknot.
- For some c in the tangle $2m_i, 2 \leq i \leq n - 1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 5** (**b**) which is unknot.
- For some c in the tangle $2m_n$, by switching $V \setminus \{c\}$ crossings, we get **Figure 5** (**c**) which is unknot.

Therefore, it was concluded that every subset of the set V with cardinality $|\ V\ |$-1 is an unknotting set. Moreover, no set with the cardinality less than $|\ V\ \ |$-1 is an unknotting set for the knot $C(2m_1 + 1, 2m_2, 2m_3, \ldots, 2m_n)$ because this knot is an alternating knot consisting of an odd tangle and n-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result, it is necessary to switch at least | V |-1 crossings to get unknot. The set V is a fault-tolerant unknotting set for $C(2m_1+1, 2m_2, 2m_3, \dots, 2m_n)$ because:

- V is an unknotting set as by switching all crossings in V, we get unknot (see **Figure 4**)
- V is a fault-tolerant unknotting set because $V \setminus \{c\}$ is an unknotting set for every $c \in V$.

Consequently, V is fault-tolerant unknotting set for $C(2m_1+1, 2m_2, 2m_3, \dots, 2m_n)$.

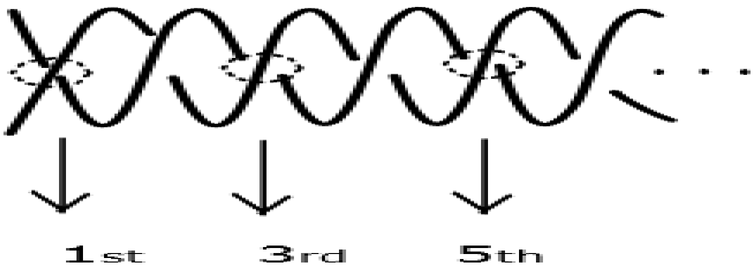

Figure 3. Choosing alternating crossings to be switched

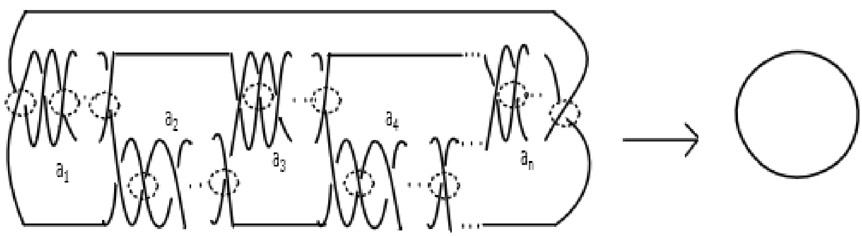

Figure 4. Switching circled crossings yields unknot

Case 2: When n is even

For even n, we have two-bridge knot of the form shown in **Figure 2**.

Theorem 3.2. The set V consisting of alternating crossings (see **Figure 3**) in $C(2m_1+1, 2m_2, 2m_3, \dots, 2m_n)$ such that:

$$|V| = \frac{(2m_1+1)+1}{2} + \frac{2m_2}{2} + \frac{2m_3}{2} + \cdots + \frac{2m_n}{2}$$

is a fault-tolerant unknotting set of $C(2m_1+1,2m_2,2m_3,\dots,2m_n)$.

Proof: Firstly, we need to prove that every subset of the set V with cardinality $|V|$-1 is an unknotting set. There are following possibilities:

- For some c in the tangle $2m_1+1$, by switching $V\setminus\{c\}$ crossings, it was obtained from **Figure 5** (**e**) which is unknot.
- For some c in the tangle $2m_i, 2\le i\le n-1$, by switching $V\setminus\{c\}$ crossings, it was obtained from **Figure 5** (**f**) which is unknot.
- For some c in the tangle $2m_n$, by switching $V\setminus\{c\}$ crossings, it was obtained from **Figure 5** (**d**) which is unknot.

Hence, it was concluded that every subset of the set V with cardinality $|V|$-1 is an unknotting set. Moreover, no set with the cardinality less than $|V|$-1 is an unknotting set for the knot $C(2m_1+1,2m_2,2m_3,\dots,2m_n)$ because this knot is an alternating knot consisting of an odd tangle and *n*-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result, it is necessary to switch at least $|V|$-1 crossings to get unknot. The set V is a fault-tolerant unknotting set for $C(2m_1+1,2m_2,2m_3,\dots,2m_n)$ because:

- V is an unknotting set as by switching all crossings in V, we get unknot (see **Figure 5**(**d**))
- V is a fault-tolerant unknotting set because $V\setminus\{c\}$ is an unknotting set for every $c\in V$

Consequently, V is fault-tolerant unknotting set for $C(2m_1+1,2m_2,2m_3,\dots,2m_n)$.

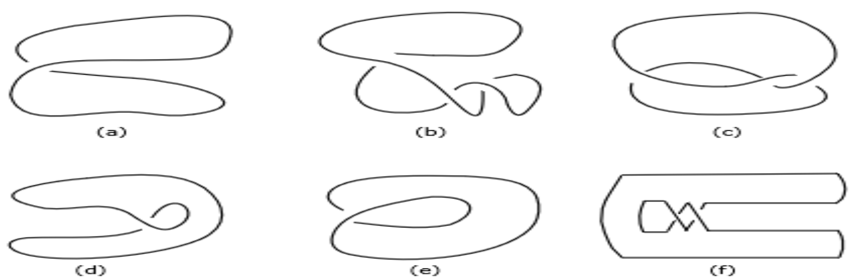

Figure 5. Unknots

3.2 The Family $C(a_1, a_2, a_3, \dots, a_n)$ with a_1, a_2 Even, a_3 Odd and $a_i's$ Even for All $3 < i \leq n$

Here, two-bridge knots of the form $C(2m_1, 2m_2, 2m_3 + 1, 2m_4, \dots, 2m_n)$ for some integers $m_1, m_2, m_3, \dots, m_n$.For n, we have following cases:

Case 1: When n is odd

For odd n, we have two-bridge knot of the form shown in **Figure 1**.

Theorem 3.3. The set V consisting of alternating crossings (see **Figure 3**) in $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \dots, 2m_n)$ such that:

$$|V| = \frac{2m_1}{2} + \frac{2m_2}{2} + \frac{(2m_3 + 1) + 1}{2} + \cdots + \frac{2m_n}{2}$$

Is a fault-tolerant unknotting set of $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \dots, 2m_n)$.

Proof: Firstly, we need to prove that every subset of the set V with cardinality $|V|$-1 is an unknotting set. For this we have following possibilities:

- For some c in the tangle $2m_1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**b**) which is unknot.
- For some c in the tangle $a_i, 2 \leq i \leq n-1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**c**) which is unknot.
- For some c in the tangle $2m_n$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**b**) which is unknot.

By above arguments, it is concluded that every subset of the set V with cardinality $|V|$-1 is an unknotting set. Moreover, no set with the cardinality less than $|V|$-1 is an unknotting set for the knot $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \dots, 2m_n)$ because this knot is an alternating knot consisting of an odd tangle and *n*-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result it is necessary to switch at least $|V|$-1 crossings to get unknot. The set V is a fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \dots, 2m_n)$ because:

- V is an unknotting set as by switching all crossings in V, we get unknot (see **Figure 6** (**a**))
- V is fault-tolerant unknotting set as $V \setminus \{c\}$ is an unknotting for every $c \in V$

- Thus, V is fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3 + 1, 2m_1 \ldots, 2m_n)$.

Case 2: When n is even

For even n, we have a two-bridge knot of the form shown in **Figure 2**.

Theorem 3.4. The set V consisting of alternating crossings (see **Figure 3**) in $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \ldots, 2m_n)$ such that:

$$|V| = \frac{2m_1}{2} + \frac{2m_2}{2} + \frac{(2m_3 + 1) + 1}{2} + \cdots + \frac{2m_n}{2}$$

is a fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3 + 1, 2m_4 \ldots, 2m_n)$.

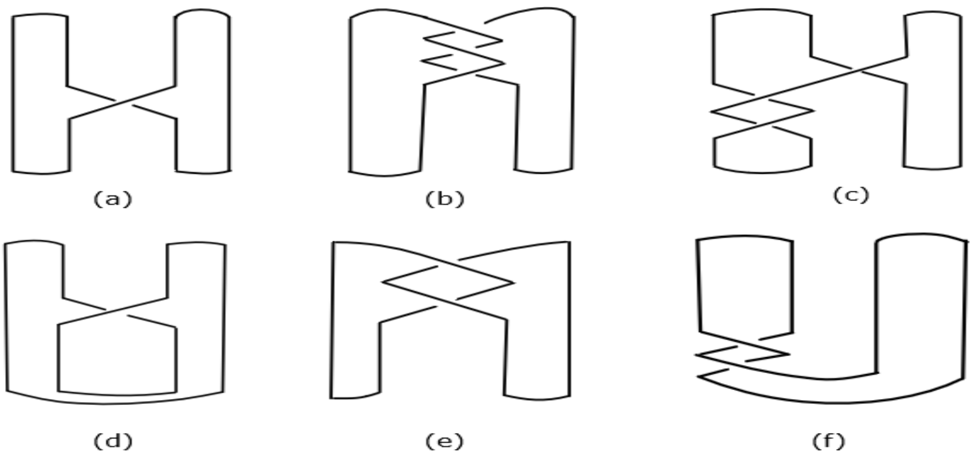

Figure 6. Unknots

Table 1: Number of $a_i's$ after a_1, a_2, a_3 for even and odd values of n

N	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
4	E	O	O	E	…	…	…	…
5	E	O	O	E	E	…	…	…
6	E	O	O	E	E	E	…	…
7	E	O	O	E	E	E	E	…
8	E	O	O	E	E	E	E	E

Proof: Firstly, it is required to prove that every subset of the set V with cardinality $|V|$-1 is an unknotting set. For this we have following possibilities:

- For some c in tangle $2m_1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**e**) which is unknot.
- For some c in tangle $a_i, 2 \leq i \leq n-1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**d**) which is unknot.
- For some $c\ in\ tangle\ 2m_n$, by switching $V \setminus \{c\}$ crossings, we get **Figure 6** (**f**) which is unknot.

By above arguments, it was concluded that every subset of the set V with cardinality $|V|$-1 is an unknotting set. Moreover, no set with the cardinality less than $|V|$-1 is an unknotting set for the knot $C\ (2m_1, 2m_2, 2m_3 + 1{,}2m_4 \ldots{,}2m_n)$ because this knot is an alternating knot consisting of an odd tangle and *n*-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result, it was necessary to switch at least $|V|$-1 crossings to get unknot. The set V is a fault-tolerant unknotting set for $C\ (2m_1, 2m_2, 2m_3 + 1{,}2m_4 \ldots{,}2m_n)$ because:

- V is an unknotting set as by switching all crossings in V, we get unknot (see **Figure 6** (**a**)).
- V is fault-tolerant unknotting set as $V \setminus \{c\}$ is an unknotting for every $c \in V$.

Thus, V is fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3 + 1{,}2m_1 \ldots{,}2m_n)$.

3.3 The Family $C(a_1, a_2, a_3, \ldots, a_n)$ with a_1 Even, a_2, a_3 Odd and $a_i's$ Even for All $4 \leq i \leq n$

Here, we take two-bridge knots of the form $C(2m_1, 2m_2 + 1{,}2m_3 + 1{,}2m_4, \ldots{,}2m_n)$ for some integers $m_1, m_2, m_3, \ldots, m_n$. For n, we have following cases:

Case 1: When n is odd

For odd n, we have two-bridge knot of the form shown in **Figure 1**. For odd n, number of $a_i's$ after a_1, a_2, a_3 must be even. This can be seen in **Table 1** for odd values of n. In **Table 1**, E stands for even and O stands for odd. We choose set V consisting of crossings chosen in the following manner:

In a_1, choose first two consecutive crossings and then alternating crossings as shown in **Figure 7** (**a**).

In $a_i, 2 \leq i \leq n-1$, choose alternating crossings in sequence with a_1 as shown in **Figure 7**(**b**). In a_n, choose second and third crossing and then alternating crossings as shown in **Figure 7** (**c**).

Theorem 3.5. V choosen in the way described above is a fault-tolerant unknotting set for $C(2m_1, 2m_2+1, 2m_3+1, 2m_4 \dots, 2m_n)$

Proof: V is an unknotting set as by switching all crossings in V, we get unknot (see **Figure 8** (**a**)).

For V to be fault-tolerant unknotting set, $V \setminus \{c\}$ must be unknotting for every $c \in V$ and we have following possibilities:

- For $c \in 2m_1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 8** (**b**) which is unknot.
- For $c \in a_i, 2 \leq i \leq n-1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 8** (**c**) which is unknot.
- For $c \in 2m_n$, by switching $V \setminus \{c\}$ crossings, we get **Figure 8** (**d**) which is unknot.

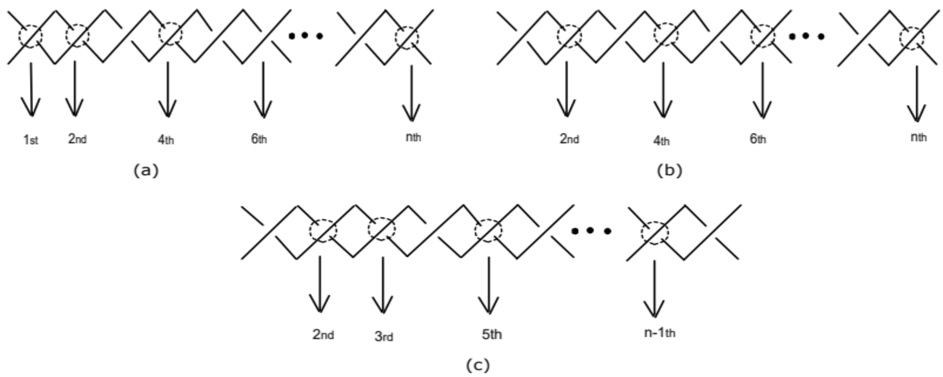

Figure 7. (a) In a_1, circled crossings switched (b) In a_2 to a_{n-1}, circled crossings switched (c) In a_n, circled crossings switched

In all possibilities, $V \setminus \{c\}$ is an unknotting set. By above arguments, we concluded that every subset of the set V with cardinality $|V|$-1 is an unknotting set. Moreover, no set with the cardinality less than $|V|$-1 is an unknotting set for the knot $C(2m_1, 2m_2 + 1,2m_3 + 1,2m_4 \ldots ,2m_n)$ because this knot is an alternating knot consisting of an odd tangle and n-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result it is necessary to switch at least $|V|$-1 crossings to get an unknot. Consequently, V is a fault-tolerant unknotting set for $C(2m_1, 2m_2 + 1,2m_3 + 1,2m_4 \ldots ,2m_n)$.

Case 2: When n is even

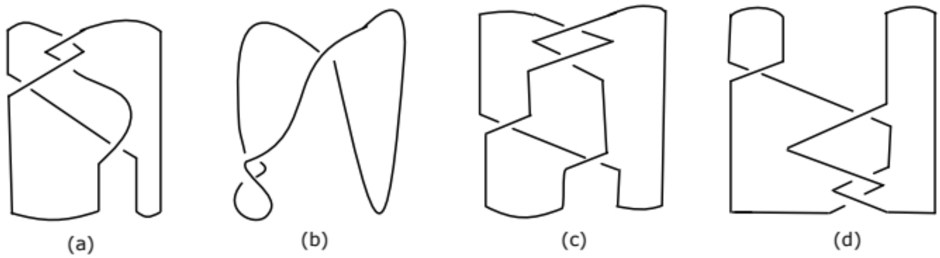

Figure 8. Unknots

For even n, we have two-bridge knot of the form shown in **Figure 2**. For even n, number of $a_i's$ after a_1, a_2, a_3 must be odd, this can be seen in the **Table 1** for even values of n. We choose a set V consisting of crossings choosen in the following manner:

In a_1, choose first two alternating crossings as shown in **Figure 9** (**a**). In a_2 and a_3, choose first two consecutive crossings and then alternating crossings as shown in **Figure 9** (**b**). In $a_i, 4 \leq i \leq n$, choose alternating crossings in sequence starting from a_4.

Theorem 3.6. V choosen in the way described above is fault-tolerant unknotting set for $C(2m_1, 2m_2 + 1,2m_3 + 1,2m_4 \ldots ,2m_n)$

Proof: V is an unknotting set as by switching all crossings in V (**Figure 10** (**a**)), we get an unknot. For V to be fault-tolerant unknotting set, $V \setminus \{c\}$ must be unknotting for every $c \in V$ and we have following possibilities.

- For $c \in 2m_1$, by switching $V \setminus \{c\}$ crossings, we get **Figure 10** (**b**) which is unknot.

- For $c \in a_2$ and a_3, by switching $V \setminus \{c\}$ crossings, we get **Figure 10** (**a**) which is unknot.

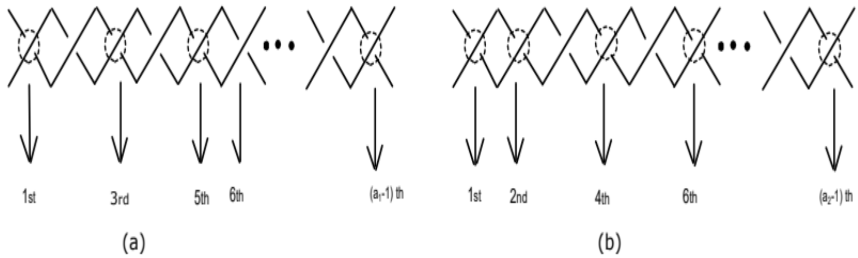

Figure 9. (a) In a_1, circled crossings switched (b) In a_2 and a_3, circled crossings switched

- For $c \in a_i, 4 \leq i \leq n$, by switching $V \setminus \{c\}$ crossings, we get **Figure 10** (**c**) which is unknot.

In all possibilities, $V \setminus \{c\}$ is an unknotting set. Consequently, V is fault-tolerant unknotting set for $C\ (2m_1, 2m_2 + 1, 2m_3 + 1, 2m_4 \ldots, 2m_n)$.

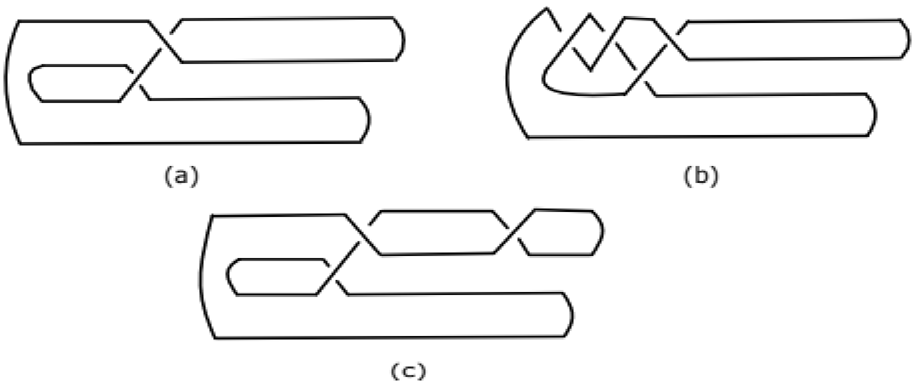

Figure 10. Unknots

3.4 The Family $C(a_1, a_2, a_3, \ldots, a_n)$ When All $a_i's$ are Even

Here, we take two-bridge knots of the form $C(2m_1, 2m_2, 2m_3, 2m_4, \ldots, 2m_n)$ for some integer$m_1, m_2, m_3, \ldots, m_n$. For n, we have following cases:

Case 1: When n is even

For even n, we have two-bridge knot of the form shown in **Figure 2**.

Theorem 3.7. A set V consisting of alternating crossings (see **Figure 3**) in $C(2m_1, 2m_2, 2m_3, \ldots, 2m_n)$ such that:

$$|V| = \frac{2m_1}{2} + \frac{2m_2}{2} + \frac{2m_3}{2} + \cdots + \frac{2m_n}{2}$$

Is a fault-tolerant tolerant unknotting set for $C(2m_1, 2m_2, 2m_3, \ldots, 2m_n)$.

Proof: Firstly, we need to prove that every subset of the set V with cardinality $|V|$-1 is an unknotting set. For this we have following possibilities:

- For some c in the tangle $2m_1$, by switching $V \setminus \{c\}$ crossings, we get unknot.
- For some c in the tangle $a_i, 2 \leq i \leq n-1$, by switching $V \setminus \{c\}$ crossings, we get unknot.
- For some c in the tangle $2m_n$, by switching $V \setminus \{c\}$ crossings, we get unknot.

With the above arguments, we concluded that every subset of the set V with cardinality $|V|$-1 is an unknotting set. Moreover, no set with the cardinality less than $|V|$-1 is an unknotting set for the knot $C(2m_1, 2m_2, 2m_3\ , 2m_4 \ldots, 2m_n)$ because this knot is an alternating knot consisting of an odd tangle and n-1 even tangles and by switching one crossing, two adjacent crossings in a tangle vanish. As a result, it is necessary to switch at least $|V|$-1 crossings to get unknot. The set V is a fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3, 2m_4 \ldots, 2m_n)$ because:

- V is an unknotting set as by switching all crossings in V, we get unknot.
- V is fault-tolerant unknotting set as $V \setminus \{c\}$ is an unknotting for every $c \in V$.

Thus, V is fault-tolerant unknotting set for $C(2m_1, 2m_2, 2m_3\ , 2m_1 \ldots, 2m_n)$.

Case 2: When n is odd

For odd n, we have two-bridge links of the form $C(2m_1, 2m_2, 2m_3, \ldots, 2m_n)$ whose fault-tolerance would be interesting to be discussed in future. The prime focus in this article is on the existence of fault-tolerant unknotting set of two-bridge knots.

3.5 The Family $C(a_1, a_2, a_3, \ldots, a_n)$ When All $a_i's$ are Odd

Here, we take two-bridge knots of the form $C(2m_1 + 1,2m_2 + 1,2m_3 + 1,2m_4 + 1, \ldots ,2m_n + 1)$ for some integers $m_1, m_2, m_3, \ldots, m_n$. For n, we have following cases:

Case 1: When $n = 2,5,8,11,14,17, \ldots$

For these values of n, we get a two-bridge link of the form $C(2m_1 + 1,2m_2 + 1,2m_3 + 1,2m_4 + 1, \ldots ,2m_n + 1\ )$.

Case 2: When $n = 1$

We have family $C(2m_1 + 1)$ for which set of alternating crossings is a fault-tolerant unknotting set.

Case 3: When $n = 3$

We have family $C(2m_1 + 1,2m_2 + 1,2m_3 + 1)$ for which set of alternating crossings is a fault-tolerant unknotting set.

Case 4: When $n = 4$

We have family $C(2m_1 + 1,2m_2 + 1,2m_3 + 1,2m_4 + 1)$ for which we have a fault-tolerant unknotting set consisting of crossings choosen in the following manner. In a_1, first two consecutive crossings then alternating crossings $\left(\frac{a_1}{2} + 1 \text{ crossings}\right)$. In a_2, a_3, starting from second crossing and then choosing alternating crossings $\left(\frac{a_2 - 1}{2} + \frac{a_3 - 1}{2} \text{ crossings}\right)$ In a_4, first two consecutive crossings then alternating crossings $\left(\frac{a_1}{2} + 1\right.$ crossings). Similarly, for other values of n, this subfamily be explored for fault-tolerance.

3.6 The Family $C(a, a_1, a_2, a_3, \ldots, a_k, \pm 2, -a_k, \ldots, -a_2, -a_1)$

In 1972, Good Rick proved that every two-bridge knot is alternating [22]. In 1986, Kanenobu and Murakami explored a subfamily of two-bridge knots shown in **Figure 11** and **Figure 12** whose unknotting number is one [23] and proved that this family has Conway notation $C(a, a_1, a_2, a_3, \ldots, a_k \pm 2, -a_k, -a_{k-1}, \ldots, -a_2, -a_1)$ [1, 18]. An analogous result for two-bridge links of unlinking number one was demonstrated by Peter Kohn [24] in 1991. Apparently, the family $C(a, a_1, a_2, a_3, \ldots, a_k, \pm 2, -a_k, -a_{k-1}, \ldots, -a_2, -a_1\ )$ of two-bridge knots is non-alternating. So, this was paradox to the result of Good Rick [22]. This apparent paradox was resolved by Nakanishi in [25] when he rearranged non-alternating diagram of this subfamily to alternating diagram by Reidemeister moves only [1].

Theorem 3.8. The weak and restricted fault-tolerant number for the family $C(a, a_1, a_2, a_3, \ldots, \mathrm{a_k}, \pm 2, -\alpha_k, -\alpha_{k-1}, \ldots, -a_2, -a_1)$ is 2.

Proof: When each crossing in the section ± 2 of $C(a, a_1, a_2, a_3, \ldots, \mathrm{a_k}, \pm 2, -\alpha_k, -\alpha_{k-1}, \ldots, -a_2, -a_1)$ is switched (not both), this two-bridge knot is untangled to unknot. So, it has two unknotting sets of cardinality 1. By theorem 1, weak fault-tolerant number of this family is 2. Furthermore, when both crossings in the section ± 2 are switched together, this knot diagram is not untangled. So, we get a restricted fault-tolerant set of cardinality 2 for this family of two-bridge knots.

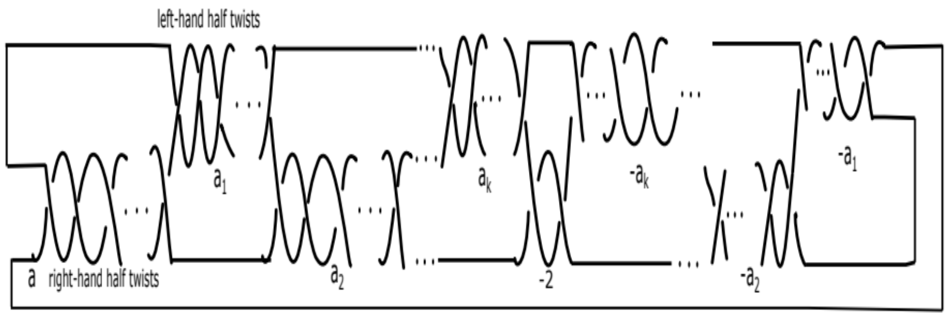

Figure 11. When k is odd

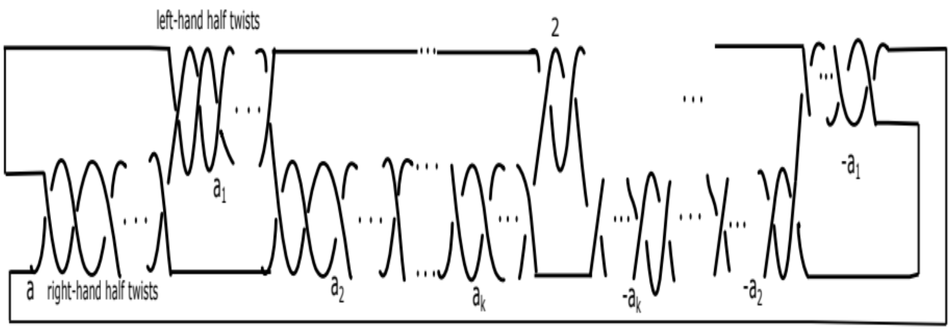

Figure 12. When k is even

3.7 The Family $C(2n, 3)$

Theorem 3.9. The family $C(2n, 3)$ of two-bridge knots has restricted fault-tolerant unknotting number $n + 1$.

Proof: $C(2n, 3)$ shown in **Figure 13** is a family of two-bridge knots. The unknotting number of $C(2n, 3)$ is n because by switching one crossing in the tangle $2n$ of $C(2n, 3)$, two adjacent crossings are destroyed. This implies that at least n crossings must be switched to destroy $2n$ crossings and consequently, the unknotting number is n. Moreover, in tangle $2n$, every set of cardinality n is an unknotting set and every set of cardinality $n + 1$ in $2n$ is a non-unknotting set. This implies that this family has restricted fault-tolerant set of cardinality $n + 1$.

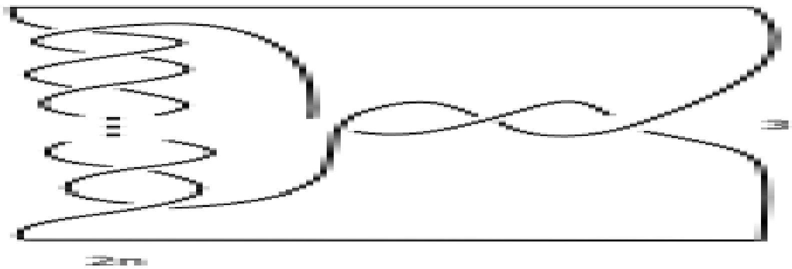

Figure 13. C (2n, 3)

4. BIOLOGICAL EXPOSITION OF FAULT-TOLERANCE IN KNOTS

The contribution of DNA topoisomerases in numerous cellular activities made their working pattern fascinating and stimulates one to explore their catalytic exertion for instance, as a successful therapeutic source in anticancer drugs [26]. Every topological readjustment of DNA for instance, unraveling of supercoiled DNA, transcription, recombination, and catenation-decatenation, initiated by un-entanglement of its complicated structure. DNA topoisomerases type-I (type-II) accomplished this task by cleaving one (two resp.) strands and permitting strands to go through cleavage before rejoining [27]. Adenosine triphosphate (ATP), manufactured in mitochondria of every cell is an energy-bank of living cells. For all key biological functions, including nerve impulse transmission, muscle contraction and protein synthesis, ATP fuels every process in cells [28]. The enzymatic activity in DNA is also ATP-based. Every time enzymes topoisomerase II act to unknot DNA, it consumes energy from ATP. The more number of enzyme actions means more energy consumption from ATP. The unknotting number refers to the least number of topoisomerase II actions to unknot DNA and least energy is consumed during enzyme action. The fault-tolerant numbers (both weak and restricted [16] may assist biologists in estimating the minimal energy transfer by ATP as well as persuade the extrication of DNA in spite of the enzyme failure at some replication fork. So, the suggested fault-tolerant numbers ensured untangling of knotted DNA at minimum energy cost bearing a miscleavage at some specific site in the DNA.

5. CONCLUSION

The fault-tolerant unknotting number takes into account the likelihood of failure of an enzyme action at some point. Explicitly, the weak fault-tolerance assured the unraveling of supercoils by subgroups of a group of enzyme actions (the group itself may or may not accomplish untangling). The restricted fault-tolerant number strictly referred to the group of enzyme actions (the group itself fails to untangle the DNA) whose subgroups are successfully untangling DNA. This unconventional interpretation of fault-tolerant numbers is speculative and referred biologists to meditate on possibilities of unknotting DNA with miscleavage by enzymes in different situations (unknotting or non-unknotting set of enzyme actions for weak fault-tolerance or necessarily non-unknotting for restricted fault-tolerance, respectively). The researchers tend to extend these computations to two-bridge links and a special subfamily of two-bridge links with unlinking number one. Moreover, some families of two-bridge knots with some special Conway notations can also be explored. Also, the complete Rolfsen Table of knots and virtual knots can be examined for fault-tolerance for the future.

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