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Cubic Intuitionistic Fuzzy Soft Set and its Distance Measures

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ABSTRACT

To deal with vagueness, falsity, attributive values, and inconsistency, this study introduced the cubic intuitionistic fuzzy soft set (CIFS-set) which is the extension of the cubic intuitionistic fuzzy set and proposed a distance measure, Hamming distance, Euclidean distance, and separation measures of CIFS-set. Moreover, we presented the aggregate operator (P-union, R-intersection) of CIFS-sets. The proposed CIFS-set is more reliable, efficient, and accurate. For the future research MCDM and MCGDM techniques could be proposed to deal with real-life issues, and this CIFS-set can also be extended for its hybrids.

Keywords: cubic set, cubic intuitionistic fuzzy soft set, fuzzy set, intuitionistic set, intuitionistic fuzzy soft set.

INTRODUCTION

Decision making is a key feature for many human activities for instance gathering information, identifying the accessing the solutions, and deliberate decisions for the relevant information. It is, therefore, not surprising that decision models are significant not just in decision theory but also in domains like engineering, operational research, management sciences, and social sciences, etc. In this era of tough competition, the primary objective of this current research is to choose the best alternative from a collection of alternatives, which would be evaluated against a variety of important factors. Therefore, decision-making (DM) techniques are a blessing for those which is decided by considering all of the friendly and unfriendly variables in their mind, but due to the increasing complications and the limitations of the human mind, they could cause extreme problems for the decision-makers, as they are unable to describe their facts with accuracy and precision. To manage this vagueness and ambiguity fuzzy set theory (FS) was initiated by Zadeh in 1965 [1, 2]. It aimed to make a rational decision in an ambiguous environment but this theory was not publicly known until Mamdani [3] applied the fuzzy logic in a practical application for a steam engine in 1974.Later, Kickert and Mamdani [4] presented the fuzzy logic controller (FLC) which converted the linguistic control strategy into an automatic control strategy, Saqlain *et al.* [5] presented the fuzzy logic controller, which was an effective approach for the representation of linguistic knowledge. In 1983, Intuitionistic fuzzy set (IF-set) theory was proposed by Atanassov [6, 7]. It was an alternative technique to sort out the uncertainties, fuzziness, and ambiguity which is characterized by the degree of membership and non-membership, as the single membership and non-membership degrees didn't deal with these scenarios precisely. Ejegwa [8] gave an overview on intuitionistic fuzzy set, De, S. K *et al.* [9] defined some operations on IF-set. After the single-valued IF-set, interval-valued IF-set [10] was introduced which is the generalization of Atanassov IF-set. In the interval-valued IF-set, the membership and non-membership functions' values are in intervals rather than exact numbers.

Zhang et al. [11] gave some information measures for interval-valued IF-set, Nayagam et al. [12] discussed the MCDM method by using interval-valued intuitionistic fuzzy set. In the above discussed methods, the information related to an element was defined either in IF-set or interval-valued IF-set which might affect the decision results. For getting precise results, a cubic intuitionistic fuzzy set (CIF-set) was introduced by Jun et al. [13] which was the extension of IF-set or interval-valued IF-set and defined the information in the form of IF-set and interval-valued IF-set simultaneously. Garg et al. [14, 15] gave the perception of cubic IF-set and discussed its fundamental properties. All these theories cannot effectively use the classical method to handle complex issues. So, the concept of soft set theory was formulated by Molodstov [16] in 1999. It is parameterized family of a subset of universal set and an extensive mathematical tool for handling the complexities and ill-defined things. Maji et al. [17] presented the Fuzzy Soft Set (FSS), a hybrid of fuzzy set and soft set, in 2001. It gave a more precise and generalized conclusion. Additionally, he came up with the concept of intuitionistic fuzzy soft set theory (IFS-set) [18] by merging the soft set theory and Intuitionistic fuzzy set (IF-set). The attributive and hesitation values obtained by IFS-soft facilitates the real-world situation more precisely, especially in multi-attributive decision making. All these theories contain the information of the belongingness and not-belonging values which played an essential role in decision-making process but these theories could not deal with situations where the decision-maker considered the truthiness and falsity value over the interval. So, it is challenging to specify the exact value in a fuzzy set corresponding to each characteristic. Thus, in the multifaceted decision-making problem, the hybrid form of interval-value and exact value could be more appropriate for a decision-maker to describe his preference. Due to this, we developed the concept of a cubic intuitionistic fuzzy soft set (CIFS-set), which concurrently specifies two components: an interval-valued intuitionistic fuzzy set (IVIF-set), and an intuitionistic fuzzy set (IF-set). Hence, the CIFS-set deals with both truthiness value and falseness value of the alternative over the interval simultaneously.

In this paper, the introduction section presents the history, literature review, and theoretical framework of the fuzzy set, intuitionistic set, soft set, fuzzy soft set, intuitionistic fuzzy soft set, cubic fuzzy set, cubic intuitionistic fuzzy set etc. In section two, some basic concepts of fuzzy, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, cubic fuzzy set, cubic intuitionistic fuzzy set, etc. are reviewed. Section three, presents the definition of cubic intuitionistic fuzzy softs set, theorems, proposition, proposed distance, normalized hamming, normalized Euclidean distance, and generalized weighted distance for different CIFS-sets. Finally, the last section concludes the current research.

2. PRELIMINARIES

In this section, we'll describe some basic concept of Fuzzy-set, Intuitionistic fuzzy set (IF-set), Soft set, Fuzzy Soft-set (FS-set), Intuitionistic fuzzy soft set (IFS-set), IVIFS-set, CFS-set, and CIFS-set.

Definition 2.1 [1]. A fuzzy set Ƒ over a universal discourse Ʊ is defined by a function $\mu_{Ƒ}$ which expresses the mapping:

$$\mu_{Ƒ} = Ʊ \rightarrow [0,1]$$

Where $\mu_{Ƒ}$ defines the membership function of Ƒ, and $\mu_{Ƒ}(ʒ)$ represents the degree of belongingness to the fuzzy set Ƒ. Thus, a fuzzy set Ƒ over set Ʊ can be described as;

$$Ƒ = \{(ʒ, \mu_{Ƒ}(ʒ)) : ʒ \in Ʊ, \mu_{Ƒ}(ʒ) \in [0,1]\}$$

Definition 2.2 [6]. Let Ʊ be a universal discourse, a function μ_I characterized by a mapping which defines an Intuitionistic fuzzy set (IF-set) Б over a crisp set Ʊ is:

$$\mu_I = \mathrm{U} \to [0,1]^2$$

Where μ_I describes the membership function which deals with truthiness and falsity value. Thus, the representation of an Intuitionistic fuzzy set (IF-set) over set universal set U;

$$Б = \{\langle ʒ, (ŧ_Б(ʒ), ʄ_Б(ʒ))\rangle | ʒ \in \mathrm{U}\} \; such\ that\ 0 \le ŧ_Б(ʒ) + ʄ_Б(ʒ) \le 1;\ \forall ʒ \in \mathrm{U}.$$

Definition 2.3 [16]. If U is the set of universes and $ɗ = \{ɗ_1, ɗ_2, ɗ_3, \ldots, ɗ_n\}$ be the set of attributes. Then, a function φ_F defined by a mapping produces a soft set Ꝃ over a universal set U:

$$\varphi_K: ɗ \to P(\mathrm{U}),$$

Where, φ_K is an approximate function and its value $\varphi_K(ɗ_i)$ could be expressed as:

$$K = \{(ɗ_i, \gamma_K(ɗ_i)): ɗ_i \in ɗ, \varphi_K(ɗ_i) \in P(\mathrm{U})\}.$$

Definition 2.4 [17]. A function φ_F represented by a mapping defines a fuzzy soft set (FS-set) Ƒ over set U is:

$$\varphi_Ƒ: ɗ \to Ƒ(\mathrm{U})\ such\ that\ \varphi_Ƒ(ɗ_i) = \emptyset\ if\ ɗ_i \notin ɗ,$$

Where, $\varphi_Ƒ$ is an approximate function, and its value $\varphi_Ƒ(ɗ_i)$ could be represented as:

$$Ƒ = \{(ɗ_i, \varphi_Ƒ(ɗ_i)): ɗ_i \in ɗ, \varphi_Ƒ(ɗ_i) \in Ƒ(\mathrm{U})\}.$$

Definition 2.5 [18]. For the purpose of dealing with parametric family of membership and non-membership values, an Intuitionistic fuzzy soft set (IFS-set) was defined. Consider $ʒ = \{ʒ_1, ʒ_2, ʒ_3, \ldots, ʒ_n\}$ be the set of alternatives and $ɗ = \{ɗ_1, ɗ_2, ɗ_3, \ldots, ɗ_n\}$ be the set of attributes. Then, a mapping defines IFS-set Ĩ over the universe discourse U is:

$$\varphi_{\tilde{I}}: ɗ \to Б(ʒ)\ such\ that\ \varphi_{\tilde{I}}(ɗ_i) = \emptyset\ if\ ɗ_i \notin ɗ,$$

Where, $\emptyset$ is the intuitionistic fuzzy empty set and $\varphi_{\tilde{I}}$ is an approximate function and its value $\varphi_{\tilde{I}}(\varkappa)$ can be defined as:

$$\varphi_{\tilde{I}}(ɗ_i) = \{\langle ɗ_i, \mu_{\varphi_{\tilde{I}}}(ɗ_i)\rangle | ɗ_i \in ɗ, \varphi_{\tilde{I}}(ɗ_i) \in \mathcal{F}(ʒ_i)\},$$

$$\tilde{I} = \{\langle ɗ_i, (ŧ_{\tilde{I}}(ʒ_i), ʄ_{\tilde{I}}(ʒ_i))\rangle | ʒ_i \in \mathrm{U}\},$$

Where, $ȶ_{\tilde{I}}$ *and* $ʄ_{\tilde{I}}$ are the mappings from Ʋ to $[0,1] \times [0,1]$ such that $0 \le ȶ_{\tilde{I}}(ʒ_i) + ʄ_{\tilde{I}}(ʒ_i) \le 1;\ \forall ʒ_i \in Ʋ$.

Definition 2.6 [19]. The concept of Intuitionistic fuzzy soft set theory (IFS-set) was extended to interval-valued intuitionistic fuzzy soft set theory (IVIFS-set) which is defined by a mapping:

$$\varphi_{Ɪ}: ɗ \rightarrow Ƃ(Ʋ)$$

$\varphi_{\tilde{I}}$ is an approximate function and its value $\varphi_{Ɪ}(ɗ_i)$ can be represented as:

$$Ɪ = \{\langle ɗ_i, \{[ȶ_{Ɪ}(ʒ_i)^-, ȶ_{Ɪ}(ʒ_i)^+], [ʄ_{Ɪ}(ʒ_i)^-, ʄ_{Ɪ}(ʒ_i)^+]\}\rangle | ʒ_i \epsilon Ʋ\}$$

Definition 2.7 [14]. A cubic fuzzy set (CF-set) Ӿ defined over Ʋ is given as:

$$Ӿ = \{\langle ɗ, \mu_{\tilde{A}}(ʒ), ν\rangle | ʒ \epsilon Ʋ\}$$

Where $\mu_{\tilde{A}}(ʒ) = [\mu_{\tilde{A}}(ʒ)^-, \mu_{\tilde{A}}(v)^+]$ *and* ν denotes the interval-valued fuzzy set and FN in $ʒ \epsilon Ʋ$ respectively. We denoted these pairs as $Ӿ = (\mu_{\tilde{A}}, ν)$ and are called cubic fuzzy set.

Definition 2.8 [20]. A cubic fuzzy soft set (CFS-set) Ӿ defined over Ʋ is given as:

$$Ӿ = \{\langle ɗ_i, \mu_{\tilde{A}}(ʒ_i), ν_i\rangle | ʒ_i \epsilon Ʋ\}$$

Where $\mu_{\tilde{A}}(ʒ_i) = [\mu_{\tilde{A}}(ʒ_i)^-, \mu_{\tilde{A}}(ʒ_i)^+]$ *and* $ν_i$ represents the interval-valued FSS and FN in $ʒ_i \epsilon Ʋ$ respectively.

We denoted these pairs as $Ӿ = (\mu_{\tilde{A}}, ν_i)$ and these pairs are called cubic fuzzy soft set.

2.1 Cubic Intuitionistic Fuzzy Soft Set (CIFS-set)

Definition 2.1.1. The term cubic intuitionistic fuzzy soft set (CIFS-set) Ӿ defined over Ʋ is stated as:

$$Ӿ = \{\langle ɗ_i, \mu(ʒ_i), ν_i\rangle | ʒ_i \epsilon Ʋ\}$$

Where $\mu(ʒ_i) = \{\langle ɗ_i, \langle [ȶ_{\tilde{A}}(ʒ_i)^-, ȶ_{\tilde{A}}(ʒ_i)^+], [ʄ_{\tilde{A}}(ʒ_i)^-, ʄ_{\tilde{A}}(ʒ_i)^+]\rangle\rangle\}$ *and* $ν_i = [ȶ_{\tilde{A}}(ʒ_i), ʄ_{\tilde{A}}(ʒ_i)]$ represents the interval-valued IFS-set and intuitionistic set

(IS) respectively such that $0 \leq \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_i) + \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_i) \leq 1, also\ 0 \leq \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_i), \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_i) \leq 1$.

We denoted these pairs as $\mathfrak{X} = (\mu_i, \mathrm{v}_i)$ and are termed as a cubic intuitionistic fuzzy soft set.

Example: Consider $\mathfrak{d} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$ be the set of attributive values and $H = \{h_1, h_2, h_3\}$ be the set of the universe and $\mathfrak{d}' = \{\mathfrak{d}_1, \mathfrak{d}_2\} \subseteq \mathfrak{d}$.

So, the CIFS-set $\mathfrak{X}$ is defined as:

$$\mathfrak{X} = \{\left\langle (\mathfrak{d}_1, \left\{\begin{matrix} (\mu_{h_1}, [0.6,0.8], [0.0,0.1], [0.7,0.1]), (\mu_{h_2}, [0.2,0.4], [0.3,0.6], [0.2,0.4]), \\ (\mu_{h_3} [0.7,0.9], [0.0,0.1], [0.8,0.0) \end{matrix}\right\}\right\rangle,$$

$$(\mathfrak{d}_2, \left\{\begin{matrix} (\mu_{h_1}, [0.6,0.8], [0.0,0.1], [0.6,0.1]), (\mu_{h_2}, [0.2,0.4], [0.1,0.3], [0.3,0.1]), \\ (\mu_{h_3} [0.2,0.7], [0.4,0.8], [0.5,0.5]) \end{matrix}\right\})\}$$

Definition 2.1.2. Let $\tilde{A} = \{\langle [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_i)^+], [\mathfrak{f}_{\tilde{A}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_i)^+] \rangle, \langle [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_i), \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_i)] \rangle\}, i = 1,2$ be two CIFS-set over $\mathbb{U}$. Then,

- **(Equality)** $\tilde{A}_1 = \tilde{A}_2 \Leftrightarrow [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^+] = [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^+], [\mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^+] = \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^+], \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1) = \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)\ and\ \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1) = \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)$
- **(P-order)** $\tilde{A}_1 \subseteq_p \tilde{A}_2 \Leftrightarrow [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^+] \subseteq [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^+], [\mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^+] \supseteq \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^+], \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1) \leq \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)\ and\ \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1) \geq \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)$
- **(R-order)** $\tilde{A}_1 \subseteq_R \tilde{A}_2 \Leftrightarrow [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1)^+] \subseteq [\mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)^+], [\mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1)^+] \supseteq \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^-, \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)^+], \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_1) \geq \mathfrak{t}_{\tilde{A}}(\mathfrak{z}_2)\ and\ \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_1) \leq \mathfrak{f}_{\tilde{A}}(\mathfrak{z}_2)$

Theorem 2.1.1. Let $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ be a CIFS-set over $\mathbb{U}$. If $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ is both ɣ-internal and ɣ-external (resp., both ɦ -internal and ɦ -external), then

$$\mathrm{v}_i(\varkappa) \in U_{\gamma}(\mu_i) \cup L_{\gamma}(\mu_i) \ \ (\text{resp.}, \mathrm{v}_i(\varkappa) \in U_{ɦ}(\mu_i) \cup L_{ɦ}(\mu_i)); \forall \varkappa \in \mathbb{U}.$$

Definition 2.1.3. Given a family $\{\mathfrak{X}_i = \langle \mu_i, \mathrm{v}_i \rangle | i \in \Lambda\}$ of CIFS-set over $\mathbb{U}$.

- $\Cup_{\rho, i \in A} \mathfrak{X}_i := \langle \bigvee_{i \in \Lambda} \mathrm{v}_i, \bigcup_{i \in \Lambda} \mu_i \rangle$ (P-union)

- $\Cup_{\mathcal{R}_{i\in A}} \mathfrak{X}_i := \langle \bigwedge_{i\in\Lambda} \mathrm{v}_i, \bigcup_{i\in\Lambda} \mu_i \rangle$ (R-union)
- $\Cap_{\rho_{i\in A}} \mathfrak{X}_i := \langle \bigwedge_{i\in\Lambda} \mathrm{v}_i, \bigcap_{i\in\Lambda} \mu_i \rangle$ (P-intersection)
- $\Cap_{\mathcal{R}_{i\in A}} \mathfrak{X}_i := \langle \bigvee_{i\in\Lambda} \mathrm{v}_i, \bigcup_{i\in\Lambda} \mu_i \rangle$ (R-intersection)

Proposition 2.1.1. For any CIFS-sets, $\mathfrak{X}_i = \langle \mu_i, \mathrm{v}_i \rangle, B_i = \langle \lambda_i, \mathcal{B}_i \rangle, C_i = \langle \mathcal{V}_i, \mathcal{C}_i \rangle$ $and\ D_i = \langle \wp_i, \mathcal{D}_i \rangle$ in Ʊ, then we have:

- $\mathfrak{X}_i \Subset_\rho B_i, \mathfrak{X}_i \Subset_\rho C_i \Rightarrow \mathfrak{X}_i \Subset_\rho B_i \Cap_\rho C_i$.
- $\mathfrak{X}_i \Subset_{\mathcal{R}} B_i, \mathfrak{X}_i \Subset_{\mathcal{R}} C_i \Rightarrow \mathfrak{X}_i \Subset_{\mathcal{R}} B_i \Cap_{\mathcal{R}} C_i$.
- $\mathfrak{X}_i \Subset_\rho C_i, B_i \Subset_\rho C_i \Rightarrow \mathfrak{X}_i \Cup_\rho B_i \Subset_\rho C_i$.
- $\mathfrak{X}_i \Subset_{\mathcal{R}} C_i, B_i \Subset_{\mathcal{R}} C_i \Rightarrow \mathfrak{X}_i \Cup_{\mathcal{R}} B_i \Subset_{\mathcal{R}} C_i$.
- $\mathfrak{X}_i \Subset_\rho B_i, C_i \Subset_\rho D_i \Rightarrow$
  $\mathfrak{X}_i \Cup_\rho C_i \Subset_\rho B_i \Cup_\rho D_i, \mathfrak{X}_i \Cap_\rho C_i \Subset_\rho B_i \Cap_\rho D_i$.
- $\mathfrak{X}_i \Subset_{\mathcal{R}} B_i, C_i \Subset_{\mathcal{R}} D_i \Rightarrow$
  $\mathfrak{X}_i \Cup_{\mathcal{R}} C_i \Subset_{\mathcal{R}} B_i \Cup_{\mathcal{R}} D_i, \mathfrak{X}_i \Cap_{\mathcal{R}} C_i \Subset_{\mathcal{R}} B_i \Cap_{\mathcal{R}} D_i$.

Theorem 2.1.2. Let $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ be a CIFS-set over Ʊ. If $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ is ɣ -internal (resp., ɣ -external), then so is the complement of $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$.

Proof: Suppose that $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ is ɣ-internal. Then,

$\mathrm{v}_i(\varkappa) \in ɣ[\mu_i](\varkappa) \Rightarrow ɣ[\mu_i]^-(\varkappa) \leq \mathrm{v}_i(\mathrm{x}) \leq ɣ[\mu_i]^+(\varkappa)\ \forall\, \varkappa \in Ʊ$

Thus; $1 - ɣ[\mu_i]^+(\varkappa) \leq 1 - \mathrm{v}_i(\varkappa) \leq 1 - ɣ[\mu_i]^-(\varkappa)$

$\Rightarrow \mathrm{v}_i{}^c(\varkappa) \in ɣ\,[\mu_i{}^c](\varkappa)\ \forall\, \varkappa \in Ʊ$

Hence $\mathfrak{X}^c = \langle \mu_i{}^c, \mathrm{v}_i{}^c \rangle$ is ɣ -internal.

Consider $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ is ɣ -external, then:

$$\mathrm{v}_i\,(\varkappa) \notin (ɣ[a]^-(\varkappa), ɣ[a]^+(\varkappa))\ \forall\, \varkappa \in Ʊ$$

And so,

$$\mathrm{v}_i(\varkappa) \in [0, ɣ[\mu_i]^-(\varkappa)]\ or\ \mathrm{v}_i(\varkappa) \in [ɣ[\mu_i]^+(\varkappa), 1]$$

It follows that

$$1 - ɣ[a]^-(\varkappa) \leq 1 - \mathrm{v}_i\,(\varkappa) \leq 1 \text{ or } 0 \leq 1 - \mathrm{v}_i\,(\varkappa) \leq 1 - ɣ[a]^+(\varkappa),$$

And so that

$$\mathrm{v}_i{}^c\,(\varkappa) = 1 - \mathrm{v}_i\,(\varkappa) \notin (1 - \gamma[\mu_i]^+\,(\varkappa), 1 - \gamma[\mu_i]^-\,(\varkappa)$$
$$= (\gamma[\mu_i{}^c]^-\,(\varkappa), \gamma[\mu_i{}^c]^+\,(\varkappa))$$

Therefore $\mathfrak{X}^c = \langle \mu_i{}^c, \mathrm{v}_i{}^c \rangle$ is γ -external.

Corollary: If a CIFS-set $\mathfrak{X} = \langle \mu_i, \mathrm{v}_i \rangle$ over U is internal, then so is $\mathfrak{X}^c = \langle \mu_i{}^c, \mathrm{v}_i{}^c \rangle$

2.2. Distance Measures for CIFS-set

In this section, we will propose the distance measure for CIFS-set.

Definition 2.2.1. Consider $\phi(\mathrm{X})$ to be the family of CIFS-set over the universal discourse $\mathrm{X} = \{x_1, x_2, x_3, \ldots, x_n\}$. Then, a mapping $\mathrm{d}: \phi(\mathrm{X}) \times \phi(\mathrm{X}) \longrightarrow [0,1]$ is called a distance measure if it satisfied the following conditions for $\mathcal{U}, \mathcal{V} and\ \mathcal{W} \in \phi(\mathrm{X})$:

- $0 \leq \mathrm{d}(\mathcal{U}, \mathcal{V}) \leq 1;$
- $\mathrm{d}(\mathcal{U}, \mathcal{V}) = 0\ iff\ \mathcal{U} = \mathcal{V};$
- $\mathrm{d}(\mathcal{U}, \mathcal{V}) = \mathrm{d}(\mathcal{V}, \mathcal{U});$
- $If\ \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}, then\ \mathrm{d}(\mathcal{U}, \mathcal{V}) \leq \mathrm{d}(\mathcal{U}, \mathcal{W})\ and\ \mathrm{d}(\mathcal{V}, \mathcal{W}) \leq \mathrm{d}(\mathcal{U}, \mathcal{W})$

Definition 2.2.2. Consider $\mathcal{U} = \{\langle \mathrm{d}_i, \langle [\mathfrak{t}_\mathcal{U}(\mathfrak{z}_i)^-, \mathfrak{t}_\mathcal{U}(\mathfrak{z}_i)^+], [\mathfrak{f}_\mathcal{U}(\mathfrak{z}_i)^-, \mathfrak{f}_\mathcal{U}(\mathfrak{z}_i)^+] \rangle, \langle [\mathfrak{t}_\mathcal{U}(\mathfrak{z}_i), \mathfrak{f}_\mathcal{U}(\mathfrak{z}_i)] \rangle\}, \mathcal{V} = \{\langle \mathrm{d}_j, \langle [\mathfrak{t}_\mathcal{V}(\mathfrak{z}_i)^-, \mathfrak{t}_\mathcal{V}(\mathfrak{z}_i)^+], [\mathfrak{f}_\mathcal{V}(\mathfrak{z}_i)^-, \mathfrak{f}_\mathcal{V}(\mathfrak{z}_i)^+], \langle \mathfrak{t}_\mathcal{V}(\mathfrak{z}_i), \mathfrak{f}_\mathcal{V}(\mathfrak{z}_i) \rangle \rangle\}$ be the two CIFS-sets, then, for $q \geq 1$, the distance measured is defined as;

- Distance measures:

$$\mathrm{d}_q(\mathcal{U}, \mathcal{V}) = \left(\langle \tfrac{1}{6} \textstyle\sum_{i=1}^{n} \{|\mathfrak{t}_\mathcal{U}(\mathfrak{z}_i)^- - \mathfrak{t}_\mathcal{V}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_\mathcal{U}(\mathfrak{z}_i)^+ - \mathfrak{t}_\mathcal{V}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_\mathcal{U}(\mathfrak{z}_i)^- - \mathfrak{f}_\mathcal{V}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_\mathcal{U}(\mathfrak{z}_i)^+ - \mathfrak{f}_\mathcal{V}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_\mathcal{U}(\mathfrak{z}_i) - \mathfrak{t}_\mathcal{V}(\mathfrak{z}_i)|^q + |\mathfrak{f}_\mathcal{U}(\mathfrak{z}_i) - \mathfrak{f}_\mathcal{V}(\mathfrak{z}_i)|^q\}\rangle\right)^{\frac{1}{q}} \quad (1)$$

- Normalized Distance measures:

$$\mathrm{d}_q'(\mathcal{U},\mathcal{V}) = \Big(\langle\frac{1}{6n}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q\}\rangle\Big)^{\frac{1}{q}} \tag{2}$$

Where n is the mean of attributive values.

Theorem 2.2.1. The measure d_q' between the two CIFS-set $\mathcal{U}\ and\ \mathcal{V}$ satisfies the conditions defined in Definition 1.

Proof: In order to prove the measure between the two CIFS-sets $\mathcal{U}\ and\ \mathcal{V}$, we shall prove that it satisfies the conditions defined in definition 3.5.

Consider $\mathcal{U} = \{\langle \mathfrak{d}_i, \langle[\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+], [\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+]\rangle, \langle[\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i), \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)]\rangle\},$

$\mathcal{V} = \{\langle \mathfrak{d}_j, \langle[\mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+], [\mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+], \langle\mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i), \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)\rangle\rangle\}$ be the two CIFS-sets.

- For any real number $q \geq 1$, the distance measured is defined as.

$$\mathrm{d}_q'(\mathcal{U},\mathcal{V}) \geq 0:$$

For an arbitrary CIFS-sets $\mathcal{U}\ and\ \mathcal{V}$, it is enough to show that $\mathrm{d}_q'(\mathcal{U},\mathcal{V}) \leq 1$.

Since $\mathcal{U}\ and\ \mathcal{V}$ are two CIFS-set, so we have;

$0 \leq \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ \leq 1$, $\quad 0 \leq \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i), \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) \leq 1,$

$0 \leq \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+, \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+ \leq 1,$ $\quad 0 \leq \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i), \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) \leq 1.$

$\Longrightarrow 0 \leq |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q \leq 1$, $0 \leq |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q \leq 1$, and

$0 \leq |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q \leq 1$, $\quad 0 \leq |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q \leq 1.$

`Similarly, $0 \leq |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q \leq 1\ and\ 0 \leq |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)|^q \leq 1$. Thus, it follows that $0 \leq \mathrm{d}_q'(\mathcal{U},\mathcal{V}) \leq 1$.

- $\mathrm{d}_q'(\mathcal{U},\mathcal{V}) = 0$:

$$\Leftrightarrow \langle\frac{1}{6n}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q\}\rangle = 0$$

$$\Leftrightarrow |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q = 0,\ |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q = 0, |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q = 0, |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q = 0, |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q = 0\ and\ |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q = 0;\ \forall\ i$$

$$\Leftrightarrow \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- = \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ = \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- = \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ = \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+, \quad \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) = \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)\ and\ \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)\ ;\ \forall\ i$$

$$\Leftrightarrow \mathcal{U} = \mathcal{V}$$

- $\mathrm{d}'_q(\mathcal{U},\mathcal{V}) = \mathrm{d}'_q(\mathcal{V},\mathcal{U})$:

For any two real numbers, a and b, we have $|a-b| = |b-a|$. Thus, we have $\mathrm{d}'_q(\mathcal{U},\mathcal{V}) = \mathrm{d}'_q(\mathcal{V},\mathcal{U})$

- $If\ \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}, then\ \mathrm{d}(\mathcal{U},\mathcal{V}) \le \mathrm{d}(\mathcal{U},\mathcal{W})\ and\ \mathrm{d}(\mathcal{V},\mathcal{W}) \le \mathrm{d}(\mathcal{U},\mathcal{W})$:

$If\ \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$ is R-order CIFS-sets then $\forall\ i$, we have;

$[\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+] \subseteq [\mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+] \subseteq [\mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^-, \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^+]$,

$[\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+] \supseteq [\mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+] \supseteq [\mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^-, \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^+]$,

$\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) \ge \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i) \ge \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)\ and\ \mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) \le \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i) \le \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)$.

Therefore,

$$|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q \le |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^-|^q, |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q \le |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^+|^q,$$

$$|\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q \ge |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^-|^q, |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q \ge |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^+|^q,$$

$$|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q \ge |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)|^q, and\ |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q \le |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)|^q.$$

Thus,

$$\mathrm{d}'_q(\mathcal{U},\mathcal{W}) = \left[\frac{1}{6n}\sum_{i=1}^{n}\{|t_{\mathcal{U}}(z_i)^- - t_{\mathcal{W}}(z_i)^-|^q + |t_{\mathcal{U}}(z_i)^+ - t_{\mathcal{W}}(z_i)^+|^q + |f_{\mathcal{U}}(z_i)^- - f_{\mathcal{W}}(z_i)^-|^q + |f_{\mathcal{U}}(z_i)^+ - f_{\mathcal{W}}(z_i)^+|^q + |t_{\mathcal{U}}(z_i) - t_{\mathcal{W}}(z_i)|^q + |f_{\mathcal{U}}(z_i) - f_{\mathcal{W}}(z_i)|^q\}\right]^{1/q}$$

$$\geq \left[\frac{1}{6n}\sum_{i=1}^{n}\{|t_{\mathcal{U}}(z_i)^- - t_{\mathcal{V}}(z_i)^-|^q + |t_{\mathcal{U}}(z_i)^+ - t_{\mathcal{V}}(z_i)^+|^q + |f_{\mathcal{U}}(z_i)^- - f_{\mathcal{V}}(z_i)^-|^q + |f_{\mathcal{U}}(z_i)^+ - f_{\mathcal{V}}(z_i)^+|^q + |t_{\mathcal{U}}(z_i) - t_{\mathcal{V}}(z_i)|^q + |f_{\mathcal{U}}(z_i) - f_{\mathcal{V}}(z_i)|^q\}\right]^{1/q}$$

Hence, $\mathrm{d}'_q(\mathcal{U},\mathcal{W}) \geq \mathrm{d}'_q(\mathcal{U},\mathcal{V})$

Similarly, $\mathrm{d}'_q(\mathcal{U},\mathcal{W}) \geq \mathrm{d}'_q(\mathcal{U},\mathcal{V})$. We can prove it for P-order CIFS-sets. Hence, $\mathrm{d}'_q(q \geq 1)$ is a valid distance measure.

Theorem 2.2.2 The measures d_q satisfies the inequality $\mathrm{d}_q \leq n^{1/q}$.

Proof: For any real number $q \geq 1$, and for two CIFS-sets $\mathcal{U}\ and\ \mathcal{V}$, we have

$$|t_{\mathcal{U}}(z_i)^- - t_{\mathcal{V}}(z_i)^-|^q \leq 1\,, |t_{\mathcal{U}}(z_i)^+ - t_{\mathcal{V}}(z_i)^+|^q \leq 1$$

And so on. Therefore, we have

$$\mathrm{d}_q(\mathcal{U},\mathcal{V}) = \left[\frac{1}{6}\sum_{i=1}^{n}\{|t_{\mathcal{U}}(z_i)^- - t_{\mathcal{V}}(z_i)^-|^q + |t_{\mathcal{U}}(z_i)^+ - t_{\mathcal{V}}(z_i)^+|^q + |f_{\mathcal{U}}(z_i)^- - f_{\mathcal{V}}(z_i)^-|^q + |f_{\mathcal{U}}(z_i)^+ - f_{\mathcal{V}}(z_i)^+|^q + |t_{\mathcal{U}}(z_i) - t_{\mathcal{V}}(z_i)|^q + |f_{\mathcal{U}}(z_i) - f_{\mathcal{V}}(z_i)|^q\}\right]^{1/q}$$

$$\leq \left[\frac{1}{6}\sum_{i=1}^{n}(1+1+1+1+1+1)\right]^{1/q}$$

$\leq n^{1/q}$

Theorem 2.2.3. The measures d_q *and* d'_q satisfies the inequality $\mathrm{d}'_q \leq \sqrt[q]{\mathrm{d}'_1}$ *and* $\mathrm{d}_q \leq \sqrt[q]{\mathrm{d}_1}$.

Proof: For any real number $q \geq 1$, and for two CIFS-sets $\mathcal{U}$ *and* $\mathcal{V}$, we have

$|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q \leq |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|$, $|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q \leq |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|$. Therefore, we get

$$\mathrm{d}'_q(\mathcal{U},\mathcal{V}) = \left[\frac{1}{6n}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q\}\right]^{1/q}$$

$$\leq \left[\frac{1}{6n}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-| + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+| + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-| + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+| + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)| + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|\}\right]^{1/q}$$

$$\leq \left(\mathrm{d}'_1(\mathcal{U},\mathcal{V})\right)^{1/q}.$$

Similarly, we proved $\mathrm{d}_q \leq \sqrt[q]{\mathrm{d}_1}$.

Theorem 2.2.4. The measures d_q *and* d'_q satisfies the equation $\mathrm{d}_q = n^{1/q}\mathrm{d}'_q$.

Proof: For any real number $q \geq 1$, and for two CIFS-sets $\mathcal{U}$ *and* $\mathcal{V}$, we have

$$\mathrm{d}_q(\mathcal{U},\mathcal{V}) = \left[\frac{1}{6}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q\}\right]^{1/q}$$

$$= n^{1/q}\left[\frac{1}{6n}\sum_{i=1}^{n}\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{V}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{V}}(\mathfrak{z}_i)|^q\}\right]^{1/q}$$

$$= n^{1/q}\mathrm{d}'_q.$$

Remark: From the proposed measure, it is observed that;

- When q =1, it reduces to a normalized hamming distance measure.
- When q=2, it reduces to a normalized Euclidean distance measure.

In real life, we often have to deal with certain situations in which the CIFS sets might have a weight assigned to them. So, the considered weights $w_i(i = 1,2, \dots, n)$, where each $w_i \geq 0\ and\ \sum_{i=1}^{n} w_i = 1$, we defined the generalized weighted distance between the two CIFS-sets $\mathcal{U}\ and\ \mathcal{W}$ as follows;

$$\mathrm{d}''_q(\mathcal{U},\mathcal{W}) = \left[\frac{1}{6}\sum_{i=1}^{n} w_i\{|\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^-|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)^+|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^- - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^-|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i)^+ - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)^+|^q + |\mathfrak{t}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{t}_{\mathcal{W}}(\mathfrak{z}_i)|^q + |\mathfrak{f}_{\mathcal{U}}(\mathfrak{z}_i) - \mathfrak{f}_{\mathcal{W}}(\mathfrak{z}_i)|^q\}\right]^{1/q} \quad (3)$$

Theorem 2.2.5. The weighted distance measures $\mathrm{d}''_q\ (1 \leq q < \infty)$, defined in Eq. (3), satisfies the following conditions:

- $0 \leq \mathrm{d}''_q(\mathcal{U},\mathcal{V}) \leq 1$;

- $d''_q(\mathcal{U},\mathcal{V}) = 0 \; iff \; \mathcal{U} = \mathcal{V}$;
- $d''_q(\mathcal{U},\mathcal{V}) = d''_q(\mathcal{V},\mathcal{U})$;
- $If \; \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}, then \; d''_q(\mathcal{U},\mathcal{V}) \leq d''_q(\mathcal{U},\mathcal{W}) \; and \; d''_q(\mathcal{V},\mathcal{W}) \leq d''_q(\mathcal{U},\mathcal{W})$.

Theorem 2.2.6. The distance measures $d_q, d'_q \; and \; d''_q$ satisfies the following inequalities:

- $d'_q \leq d_q \leq \sqrt[q]{d_1}$
- $d''_q \leq d_q \leq \sqrt[q]{d_1}$

Remark: From the proposed measure, it is observed that;

- When q =1, reduces to weighted Hamming distance measure, and
- When q=2, it reduces to a weighted Euclidean distance measure.
- When $w_i = {}^{1}/_{n} \; for \; (i = 1,2,\ldots,n)$, then Eq. (3) reduces to (2).

3. CONCLUSION

In this research paper, we proposed the cubic Intuitionistic fuzzy soft set (CIFS-set), which deals with ambiguity, falsehood, attributive values, and inconsistency, which is an extension of the cubic Intuitionistic set (CIF-set) theory. Moreover, to tackle the issue of MCGDM in the CIFS-set environment, we proposed a family of distance measures based on normalized hamming, Euclidean, and separation measures that have been developed along with their desired relations. According to the previous studies, these distance measurements used in association with a decision-making technique can simulate uncertainty considerably more effective than the current approaches and provide us a thorough understanding of real-life scenarios. The findings of this current research are applicable for the future hyper-soft set, neutrosophic soft set, and for other ambiguous and uncertain environments.

REFERENCES

1. Klir GJ, Yuan B. eds. *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*; vol 6. World Scientific;1996
2. Zadeh LA. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets Syst.* 1978;1(1):3-28.

3. Mamdani EH, Assilian S. An experiment in linguistic synthesis with a fuzzy logic controller. *Int J Man-Mach Stud*. 1975;7(1):1-13. https://doi.org/10.1016/S0020-7373(75)80002-2
4. Kickert WJM, Mamdani EH. Analysis of a fuzzy logic controller. In: Dubois D, Prade H, Yager RR, ed. *Readings in fuzzy sets for intelligent systems*. Morgan Kaufmann; 1993:290-297.
5. Saqlain M, Naz K, Gaffar K, Jafar MN. Fuzzy logic controller. *Scie Inquiry Re*. 2019;3(3):16-29.
6. Atanassov K. Intuitionistic fuzzy sets. *Int J Bioautomation*. 2016;20(Suppl1):1-6.
7. Atanassov KT. *On intuitionistic fuzzy sets theory*. vol 283. Springer;2012
8. Ejegwa PA, Akowe SO, Otene PM, Ikyule JM. An overview on intuitionistic fuzzy sets. *Int J Sci Technol Res*. 2014;3(3):142-145.
9. De SK, Biswas R, Roy AR. Some operations on intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 2000;114(3):477-484. https://doi.org/10.1016/S0165-0114(98)00191-2
10. Atanassov KT. Interval valued intuitionistic fuzzy sets. In: Atanassov KT, ed. *Intuitionistic fuzzy sets*. vol 35. Heidelberg; 1999:139-177.
11. Zhang QS, Jiang S, Jia B, Luo S. Some information measures for interval-valued intuitionistic fuzzy sets. *Inf Sci*. 2010;180(24):5130-5145. https://doi.org/10.1016/j.ins.2010.08.038
12. Nayagam VLG, Muralikrishnan S, Sivaraman G. Multi-Criteria decision-making method based on interval-valued intuitionistic fuzzy sets. *Exp Syst Appl*. 2011;38(3):1464-1467. https://doi.org/10.1016/j.eswa.2010.07.055
13. Jun YB, Kim CS, Yang KO. Cubic sets. *Ann Fuzzy Math Info*. 2012;4(1):83-98.
14. Garg H, Kaur G. Cubic intuitionistic fuzzy sets and its fundamental properties. *J Mult. -Valued Log Soft Comput*. 2019;33(6):507-537
15. Kaur G, Garg H. Cubic intuitionistic fuzzy aggregation operators. *Int J Uncertain Quantif*. 2018;8(5):405-427. https://doi.org/10.1615/Int.J.UncertaintyQuantification.2018020471
16. Molodtsov D. Soft set theory—first results. *Comput Math with*. 1999;37(4-5):19-31.
17. Maji PK, Biswas RK, Roy A. Fuzzy soft sets. *Sci Info Data*. 2001;9(3):589-602.

18. Maji PK, Biswas R, Roy AR. Intuitionistic fuzzy soft sets. *J Fuzzy Math*. 2001;9(3):677-692.
19. Zulqarnain RM, Xin XL, Saqlain M, Khan WA. TOPSIS method based on the correlation coefficient of interval-valued intuitionistic fuzzy soft sets and aggregation operators with their application in decision-making. *J Math*. 2021;2021:e6656858. https://doi.org/10.1155/2021/6656858
20. Garg H, Kaur G. Extended TOPSIS method for multicriteria group decision-making problems under cubic intuitionistic fuzzy environment. *Sci Iran*. 2020;27(1):396-410. https://doi.org/10.24200/SCI.2018.5307.1194