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Shape Designing Using Quadratic Trigonometric B-spline Curves

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Abstract

The B-spline curves, particularly trigonometric B-spline curves, have attained remarkable significance in the field of Computer Aided Geometric Designing (CAGD). Different researchers have developed different interpolants for shape designing using Ball, Bezier and ordinary B-spline. In this paper, quadratic trigonometric B-spline (piecewise) curve has been developed using a new basis for shape designing. The proposed method has one shape parameter which can be used to control and change the shape of objects. Different objects like flower, alphabet and vase have been designed using the proposed method. The effects of shape parameter and control points have been discussed also.

Keywords: trigonometric B-spline curves, quadratic trigonometric polynomial, basis functions, trigonometric Bezier curves

1. Introduction

In implementing the algorithm and 3D designing of various products, Computer Aided Geometric Designing (CAGD) has played a vital role. The formal concept of CAGD emerged in 1970s and expanded to various fields, such as automobile industry and rapid prototype machines. The polynomial B-spline, particularly the quadratic and cubic B-spline, have become the most useful tools in CAGD for designing different products.

Trigonometric B-spline also plays an important role in shape designing. Schoenberg introduced the trigonometric B-spline. In 1979, Lyche and Winther established the recurrence relation for trigonometric B-spline.

Different researchers have used different techniques for different applications, such as [1, 2] used the Rational Ball curve for image reconstruction. Trigonometric Bezier curve with the addition of two shape parameters was introduced by [3]. The author of [4] worked on multivariate trigonometric B-spline. It was hard and time taking to make calculations for higher degree polynomials. To save time and reduce the number of calculations, the idea of knot insertion in trigonometric curves was presented by [5]. Different methods of interpolations were developed to approximate curves with the addition of the trigonometric function [6]. It was a hurdle in geometrical modeling to manage curves at the given control points. To cope with this problem, shape preserving properties, especially the concept of convex hull property, was given by [7]. Harmonic rational Bezier curve was related to polynomial Bezier curve in [8]. Trigonometric cubic Bezier curve was also used for the purpose of geometrical modeling by [9]. Trigonometric cubic Bezier curve was used for shape designing with two shape parameters. The trigonometric cubic Bezier curve with one shape parameter was introduced by [10]. Generalization of Bezier curve and surfaces was presented by Han in [11].

Trigonometric Bezier curves using different shape parameters have been applied in shape designing in order to control the shape of objects; however, quadratic trigonometric B-spline curves are considered to be more appropriate for shape designing. This is due to the fact that in trigonometric Bezier curves, we need more control points as compared to trigonometric B-spline curves. Trigonometric B-spline curves have local control, whereas trigonometric Bezier curves have global control. The curves can be approximated within the convex hull in a better way using shape parameters.

The aim of this paper is to introduce practical and piecewise trigonometric polynomial curves with one shape parameter. The proposed curves have been implemented to design different shapes such as English alphabet A, vase and flower. The designed shape can be controlled or changed by changing the values of shape parameter.

2. Basis Functions for the Quadratic Trigonometric Polynomial

We can construct the basis functions. Suppose $a_0 < a_1 < a_2 < \ldots < a_{n+3}$ are the knots
and $\Delta a_i = a_{i+1} - a_i$, then

$$f(u) = 1-(1+v)\sin u + v\sin^2 u,$$

$$g(u) = 1-(1+v)\cos u + v\cos^2 u, \qquad -1 \le v \le 1$$

$$C_i(a) = \begin{cases} \omega_i g(u_i), & a \in [a_i, a_{i+1}) \\ 1-\gamma_{i+1} f(u_{i+1}) - \omega_{i+1} g(u_{i+1}) & a \in [a_{i+1}, a_{i+2}) \\ \gamma_{i+2} f(u_{i+2}), & a \in [a_{i+2}, a_{i+3}) \\ 0, & \text{Otherwise} \end{cases} \qquad (1)$$

where

$$\gamma_i = \frac{\Delta a_i}{\Delta a_{i\text{-}1} + \Delta a_i}, \omega_i = \frac{\Delta a_i}{\Delta a_i + \Delta a_{i+1}}, u_i(a) = \frac{\pi}{2}\frac{a\text{-}a_i}{\Delta a_i}$$

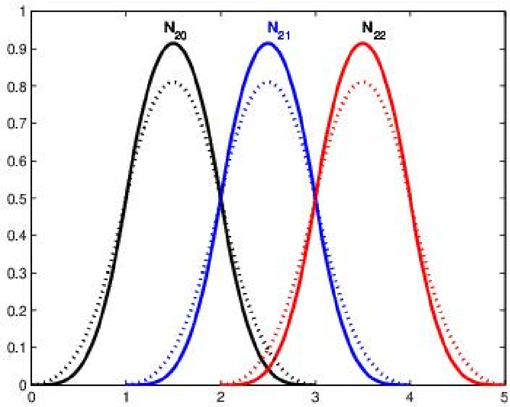

Figure 1. Trigonometric basis functions

Graphical behavior of the quadratic trigonometric basis functions is shown in figure 1.

3. Properties of the Basis Functions

Positivity

$$C_i(a) > 0, \text{for } a_i < a < a_{i+3}$$

Proof

$$\text{Let } \lambda_{i+1} = \max\{\gamma_{i+1} + \omega_{i+1}\}, \qquad \text{For } a\epsilon[a_{i+1}, a_{i+2})$$
$$\gamma_{i+1} f(u_{i+1}) + \omega_{i+1} g(u_{i+1}) \leq \lambda_{i+1}\big(f(u_{i+1}) + g(u_{i+1})\big),$$
$$= \lambda_{i+1}[(1+v)(1\text{-}(\sin u_{i+1} + \cos u_{i+1})], \leq \lambda_{i+1} < 1.$$

for $a\epsilon[a_{i+1}, a_{i+2}), i = 2,3, \dots, n.$

- **Local Support**

$$C_i(a) = 0, \text{for } a_0 < a < a_{i+3}, a_{i+3} < a < a_{n+3}.$$

- **Piecewise Polynomial:** The trigonometric B-spline have the piecewise polynomial function as defined in section 2.

- **Partition of Unity**

$$\sum_{i=1}^{k} C_i = 1, \quad a \in [a_2, a_{n+1}).$$

Proof

$$C_{i-2}(a) = \gamma_i f(u_i),$$
$$C_{i-1}(a) = 1\text{-}\gamma_i f(u_i)\text{-}\omega_i g(u_i),$$
$$C_i(a) = \omega_i g(u_i),$$
$$C_k(a) = 0, k \neq i\text{-}2, i\text{-}2, i,$$
$$\sum_{i=0}^{n} C_k(a) = C_{i-2}(a) + C_{i-1}(a) + C_i(a),$$
$$\sum_{i=0}^{n} C_k(a) = 1$$

- **Continuity:** The basis functions satisfy the continuity property at knot points. The trigonometric basis function $C_i(a)$ has C^1continuity.

3.1. Continuity at First Knot:

$$C_i(a_{i+1}^-) = \omega_i, C_i(a_{i+1}^+) = 1\text{-}\gamma_i,$$

L.H.S continuity

$$C_i(a) = \omega_i g(u_i),$$

Putting values, we have

$$\omega_i g(u_i) = \omega_i\{1\text{-}(1+v)\cos(u_i) + v\cos^2(u_i)\},$$
$$= \omega_i\left\{1\text{-}(1+v)\cos\left(\frac{\pi}{2}\frac{a\text{-}a_i}{\Delta a_i}\right) + v\cos^2\left(\frac{\pi}{2}\frac{a\text{-}a_i}{\Delta a_i}\right)\right\},$$

Replacing a by a_i, we have

$$C_i(a_{i+1}^-) = \omega_i\left\{1\text{-}(1+v)\cos\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_i}{\Delta a_i}\right) + v\cos^2\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_i}{\Delta a_i}\right)\right\},$$
$$= \omega_i\left\{1\text{-}(1+v)\cos\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) + v\cos^2\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right)\right\},$$
$$= \omega_i\{1\text{-}(1+v)\cos(\pi/2) + v\cos^2(\pi/2)\},$$
$$C_i(a_{i+1}^-) = \omega_i. \quad \text{Hence} \quad \text{proved}$$

Now, we have

$$C_i(a_{i+1}^+) = 1\text{-}\gamma_i,$$

R.H.S continuity

$$1\text{-}\gamma_{i+1} f(u_{i+1})\text{-}\omega_{i+1} g(u_{i+1}),$$

Putting the value of u_{i+1}

$$1\text{-}\gamma_{i+1}f(u_{i+1})\text{-}\omega_{i+1}g(u_{i+1})$$
$$= 1\text{-}\gamma_{i+1}\left\{1\text{-}(1\text{-}v)\sin\left(\frac{\pi}{2}\frac{a\text{-}a_{i+1}}{\Delta a_{i+1}}\right) + v\sin^2\left(\frac{\pi}{2}\frac{a\text{-}a_{i+1}}{\Delta a_{i+1}}\right)\right\},$$
$$\text{-}\omega_{i+1}\left\{1\text{-}(1\text{-}v)\cos\left(\frac{\pi}{2}\frac{a\text{-}a_{i+1}}{\Delta a_{i+1}}\right) + v\cos^2\left(\frac{\pi}{2}\frac{a\text{-}a_{i+1}}{\Delta a_{i+1}}\right)\right\},$$

Replacing a by a_{i+1}

$$= 1\text{-}\gamma_{i+1}\left\{1\text{-}(1\text{-}v)\sin\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_{i+1}}{\Delta a_{i+1}}\right) + v\sin^2\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_{i+1}}{\Delta a_{i+1}}\right)\right\},$$
$$\text{-}\omega_{i+1}\left\{1\text{-}(1\text{-}v)\cos\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_{i+1}}{\Delta a_{i+1}}\right) + v\cos^2\left(\frac{\pi}{2}\frac{a_{i+1}\text{-}a_{i+1}}{\Delta a_{i+1}}\right)\right\},$$
$$= 1\text{-}\gamma_{i+1}\{1\text{-}(1\text{-}v)\sin(0) + v\sin^2(0)\}\text{-}\omega_{i+1}\{1\text{-}(1\text{-}v)\cos(0) + v\cos^2(0)\},$$
$$= 1\text{-}\gamma_{i+1}\{1 + 0 + 0\}\text{-}\omega_{i+1}\{1\text{-}1 + v\text{-}v\},$$
$$C_i(a_{i+1}^{+}) = 1\text{-}\gamma_{i+1}. \quad \text{Hence proved}$$

Similarly,

$$C_i(a_{i+2}^{-}) = 1\text{-}\omega_{i+1}, C_i(a_{i+2}^{+}) = \gamma_{i+2},$$

and

$$C_i'(a_{i+1}^{-}) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_i}\omega_i, \quad C_i'(a_{i+1}^{+}) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_{i+1}}\gamma_{i+1},$$
$$C_i'(a_{i+2}^{-}) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_{i+1}}\omega_{i+1}, \quad C_i'(a_{i+2}^{+}) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_{i+2}}\gamma_{i+2},$$
$$\gamma_{j+1} = 1\text{-}\omega_{i+1}, \quad {\gamma_{j+1}}/{\Delta a_{j+1}} = {\omega_j}/{\Delta a_j}, \quad (0 \le j \le n+1),$$

So, we obtain the result as

$$C_i^{(k)}(a_{i+1}^{-}) = C_i^{(k)}(a_{i+1}^{+}), \quad C_i^{(k)}(a_{i+2}^{-}) = C_i^{(k)}(a_{i+2}^{+}), \quad \text{for } k = 0,1.$$

If $v = 0$ then our quadratic trigonometric B-spline basis function will be the linear trigonometric basis function.

Remarks: It is important to note that we can only discuss the continuity of the basis functions at knot points. It is obvious that continuity cannot be discussed at the first and last point.

4. Quadratic Trigonometric B-Spline Curve

For the points $p_i(i = 0,1,2,..,n)$in R^2 or R^3 and $= (a_0, a_1, a_2, \ldots, a_{n+3})$

$$S(a) = \sum_{j=0}^{n} Cj(a)p_j$$

is known as quadratic trigonometric B-spline polynomial curve with shape parameter.

4.1. The Continuity Between Two Curve Segments

If $a_i \neq a_{i+1}$ $(2 \leq i \leq n)$, the representation of the curve segment $S(a)$ can be as follows,

$$S(a) = C_{i-2}(a) + C_{i-1}(a) + C_i(a),$$

Moreover,

$$S(a_i^+) = \gamma_i p_{i-2} + (1-\gamma_i)p_{i-1},$$
$$S(a_{i+1}^-) = (1-\omega_i)p_{i-1} + \omega_i p_i,$$
$$S'(a_i^+) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_{i+1}}\gamma_i(p_{i-1}-p_{i-2}),$$
$$S'(a_{i+1}^-) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_{i+1}}\omega_i(p_i-p_{i-1}),$$

4.2. Closed and Open Trigonometric Curves

When we generate the curve $S(a)$ in the interval $[a_2, a_{n+1}]$, we are free from the choice of first and last two knots. These can be adjusted to the given boundary behavior of the curve. The choice of knot vector for open TC is as follows,

$$A = (a_0 = a_1 = a_2, a_n, a_{n+1} = a_{n+2} = a_{n+3})$$

This results show that the points p_0 and p_n are points in the curve and interior knots can be multiple knots.

$$S'(a_2^+) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_2}(p_1-p_0), \quad S'(a_{n+1}^-) = \frac{\pi}{2}\frac{(1+v)}{\Delta a_2}(p_n-p_{n-1}),$$

$$C_{n-1}(a) = \begin{cases} \omega_{n-1}g(u_{n-1}), & a \in [a_{n-1}, a_n) \\ 1-\gamma_n f(u_n)-\omega_n g(u_n), & a \in [a_n, a_{n+1}) \\ \gamma_0 f(u_0), & a \in [a_0, a_1) \\ 0, & \text{Otherwise} \end{cases} \qquad (2)$$

$$C_n(a) = \begin{cases} \omega_n g(u_n), & a \in [a_n, a_{n+1}) \\ 1-\gamma_0 f(u_0)-\omega_n g(u_0), & a \in [a_0, a_1) \\ \gamma_1 f(u_1), & a \in [a_1, a_2) \\ 0, & \text{otherwise} \end{cases} \qquad (3)$$

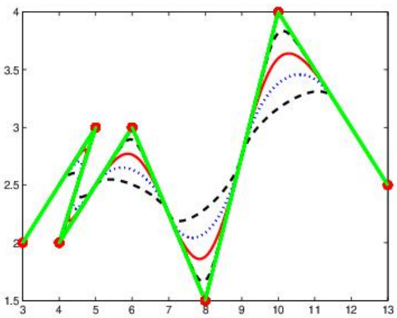

Figure 2. Open curve with different values of shape parameter

Table 1. Behavior of the Open Curve figure (2) with Different Colors

Sr. no.	Curve	value of v
1	Black	1
2	Red	0.5
3	Blue	0
4	Black	-0.5

Different values of v like $v = 1, v = 0.5, v = 0, v = -0.5$ have been used for the construction of quadratic trigonometric B-spline curve as shown in figure 1. It is observed that by increasing the value of v, the curve moves toward the control polygon. Table 1 shows the different values of v.

5. Implementation of the Proposed Method

In this section, we have constructed three different objects, that is, English alphabet A, vase and flower.

Example 1: We can control the shape of English alphabet A in two ways. Firstly, with the help of shape parameter and secondly, by inserting more control points.

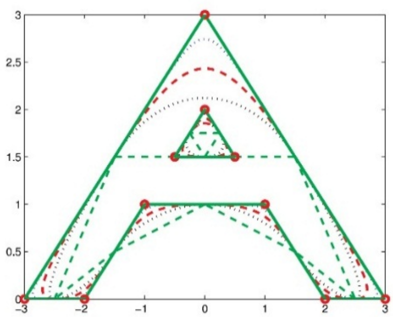

Figure 3 (a). Effects of shape parameter

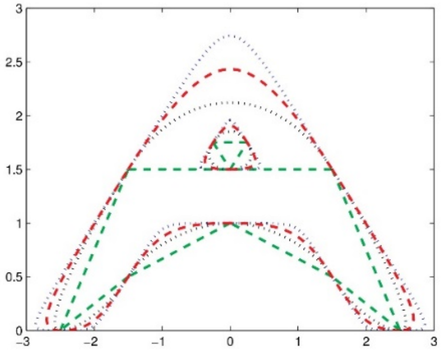

Figure 3 (b). Different values of shape parameter control points

Table 2. Effect of Shape Parameter with Different Colored Curves {Figures (3a-3b)}

Sr.no	Curve	Value of v	Value of γ	Value of ω
1	Blue	1	1/2	1/2
2	Red	0.5	1/2	1/2
3	black	0	1/2	1/2
4	green	-1	1/2	1/2

The English alphabet A has been constructed using quadratic trigonometric B-spline with different shape parameters, as shown in figure 3(a) with the help of control points. The design of alphabet A can be changed with the help of shape parameters as shown in figure 3(b). The different values of shape parameters used in figure 3a and 3b are shown in table 2.

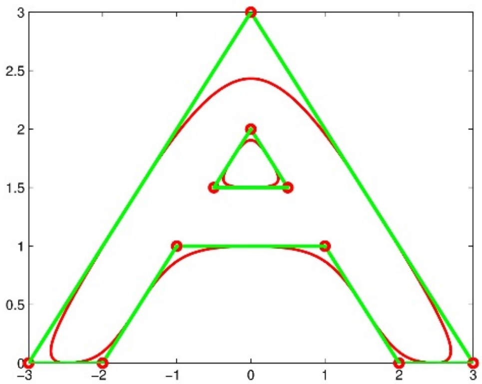

Figure 4a. Image with 10 control points

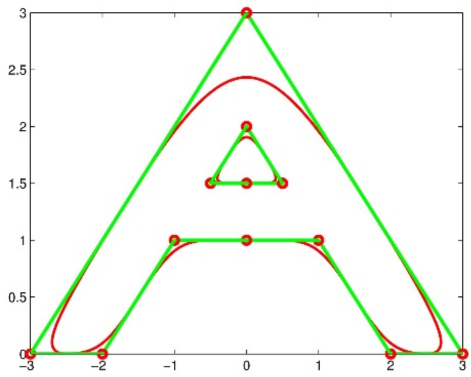

Figure 4b. Image with 12 control points

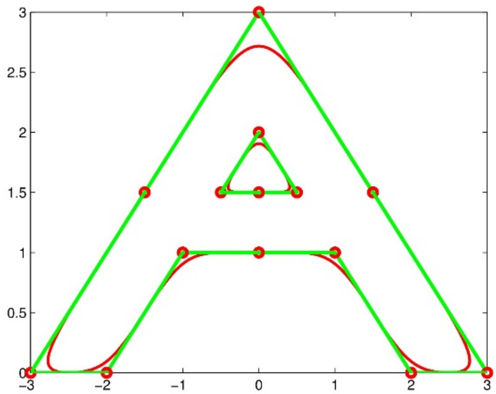

Figure 4c. Image with 16 control points

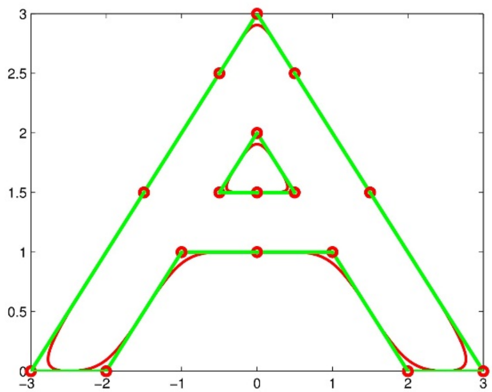

Figure 4d. Image with 20 control points

The effect of control points is shown in figure 4. Table 3 shows the number of control points used in figures 4a-4d.

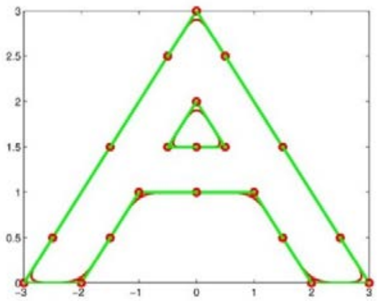

Figure 4e. Image with 22 control points

Table 3. Effect of Knot Insertion in Font Designing using Different Colors is Shown in Figures 4a-4d.

Sr. no.	Curve	No. of control points
1	fig a	10
2	fig b	12
3	fig c	16
4	fig d	20
4	fig e	22

Example 2

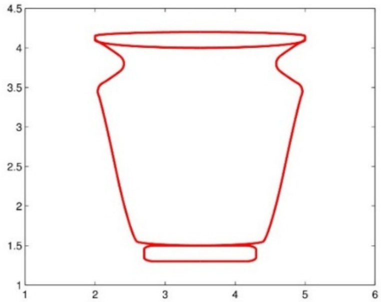

Figure 5a. Vase designing

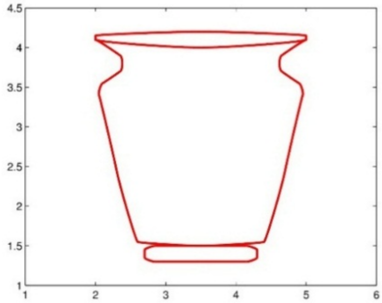

Figure 5b. Vase designing

In example 2, the shape of vase has been constructed using quadratic trigonometric B-spline by taking $v = -0.5$ as shown in figure 5(a) and $v = 0$ as shown in figure 5(b).

Example 3

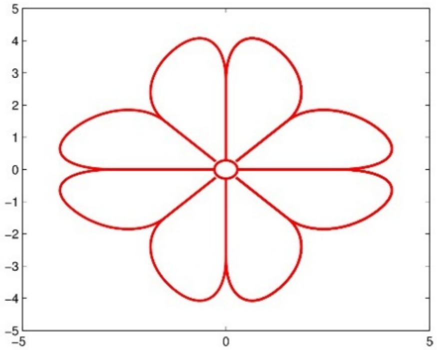

Figure 6a. Flower designing

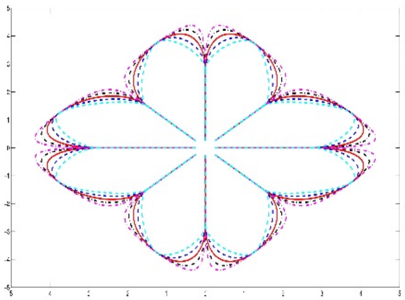

Figure 6b. Flower designing

Note: We can relate quadratic B-spline as

$$B_k(t) = b_{k0}(v) + b_{k1}(v) + b_{k2}(v)$$
$$t = \frac{a\text{-}a_k}{\Delta a_k}$$
$$b_{k0} = \gamma_k(1\text{-}t)^2$$
$$b_{k1} = 1\text{-}\gamma_k(1\text{-}t)^2\text{-}\omega t^2$$
$$b_{k2} = \omega_k t^2$$

Theorem: Let P_{k-2}, P_{k-1}, P_k be not collinear $R_{k-1} = \lambda P_{k-2} + (1\text{-}\lambda)P_{k-2}$, then there exist unique v^* and u^*, such that $B_k(v^*)$ and $S_k(u^*)$ are the intersection points of the line segment $P_{k-1}R_{k-1}$ with the curve $B_k(v)$ and $S_k(u)$, respectively for $a \in [a_k, a_{k-1}]$. Now,

$$S_k(u^*)\text{-}P_{k-1} = h(u^*)(B_k(v^*)\text{-}P_{k-1})$$

$$h(u^*) = (f(u) + g(u))$$

Proof

$$B_k(v) = P_{k\text{-}1} + \left(b_{k0}(v) + b_{k2}(v)\right)\left(\frac{b_{k0}(v)P_{k\text{-}2} + b_{k2}(v)\, P_k}{b_{k0}(v) + b_{k2}(v)} \text{-}P_{k\text{-}1}\right)$$

and

$$S_k(u) = P_{k\text{-}1} + \left(c_{k0}(u) + c_{k2}(u)\right)\left(\frac{c_{k0}(u)P_{k\text{-}2} + c_{k2}(u)\, P_k}{c_{k0}(u) + c_{k2}(u)} \text{-}P_{k\text{-}1}\right)$$

where

$$\left(\frac{b_{k0}(v)P_{k\text{-}2} + b_{k2}(v)\, P_k}{b_{k0}(v) + b_{k2}(v)}\right), \qquad \left(\frac{c_{k0}(u)P_{k\text{-}2} + c_{k2}(u)\, P_k}{c_{k0}(u) + c_{k2}(u)}\right)$$

are the points on $P_{k\text{-}1}, p_k$ and

$$\left(\frac{b_{k0}(v)}{b_{k0}(v) + b_{k2}(v)}\right), \qquad \left(\frac{c_{k0}(u)}{c_{k0}(u) + c_{k2}(u)}\right). \quad (\text{-}1 < v \leq 1)$$

are monotones for $v \in [0,1], u \in [0, \pi/2]$ and there exist v^* and u^* such that

$$\frac{b_{k0}(v^*)}{b_{k0}(v^*) + b_{k2}(v^*)} = \frac{c_{k0}(u^*)}{c_{k0}(u^*) + c_{k2}(u^*)} = \lambda$$

which results in

$$b_{k0}(v^*)c_{k2}(u^*) = b_{k2}(v^*)c_{k0}(u^*)$$

so

$$v^{*2}f(u^*) = (1\text{-}v^*)g(u^*)$$
$$f(u^*) = (1\text{-}v^*)h(u^*).$$
$$g(u^*) = v^{*2}h(u^*).$$
$$c_{k0}(u^*) + c_{k2}(u^*) = \left(b_{k0}(v^*) + b_{k2}(v^*)\right)h(u^*).$$

Hence proved.

6. Conclusion

The quadratic trigonometric B-spline curve with one shape parameter has been derived in this paper. The quadratic trigonometric B-spline curve satisfies basic properties like convex hull, affine invariant etc.

Derived B-spline curves have been employed to design different shapes such as font 'A', vase and flower. The shape parameter is very helpful in designing different objects. The user can control and change the shape using the free shape parameter. The quadratic trigonometric B-spline curve gives batter results as compared to trigonometric Bezier curves. Also, the computational cost is cheaper as compared to trigonometric Bezier curves.

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