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Recent Developments regarding Lacunary Δ-Statistical Convergence in Neutrosophic n-Normed Linear Spaces

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ABSTRACT

The current paper's goal is to introduce lacunary Δ-statistically convergent and lacunary Δ-statistically Cauchy sequences in neutrosophic n-normed linear spaces, examine them, and come to some significant conclusions about them. Also, we prove several useful results for these notions. We demonstrate how sequences in this space have some characteristics with lacunary Δ-statistical convergence of real sequences. In contrast to other related works, we define the concept of convergence of a sequence in neutrosophic n-normed linear spaces in this study. We have also established the results using a new methodology. Additionally, we provide some novel descriptions for lacunary sequences that are Δ-statistically convergent and Cauchy. We demonstrate some inclusion results between the set of Δ-statistically convergent and lacunary Δ-statistically convergent sequences in neutrosophic n-normed linear spaces by generalizing the concepts for complex number sequences.

Keywords: cauchy sequence, lacunary sequence, normed space, statistical convergence

INTRODUCTION

In 1965, Zadeh [1] introduced the fuzzy set theory. This theory has been used in numerous areas of mathematics, including metric and topological spaces.This theory further develops as the theory of functions, and approximation theory, in addition to other numerous sectors of engineering, such as population dynamics, nonlinear dynamic systems, and quantum physics. Researchers Kim and Cho [2], Malceski [3], and Gunawan and Mashadi [4] all investigated n-normed linear spaces. Fuzzy n-normalized linear space which was defined by Vijayabalaji and Narayanan [5]. Schweizer [6] introduced the continuous t-norm and Atanassov [7] used continuous t-norm and continuous t-conorm to introduce the intuitionistic fuzzy sets. Intuitionistic fuzzy normed space was first introduced by Saadati and Park [8], while intuitionistic fuzzy n-normed space was defined by Vijayabalaji et al. [9]. The concept of lacunary statistical convergence was first proposed by Fridy and Orhan [10].This concept became the basis for the investigations of lacunary statistical convergence by Mursaleen and Mohiuddin [11] and Sen and Debnath [12] in intuitionistic fuzzy normed spaces and intuitionistic fuzzy n-normed spaces, respectively. Kizmaz developed the concept of difference sequences, where $\Delta x = (\Delta x_k) = x_k - x_{k+1}$. The Δ-Statistical convergence of sequences was first described by Basarir [13]. The definition of lacunary strong -convergence of fuzzy numbers was first developed by Bilgin [14]. Many authors have also researched the generalized difference between the sequence spaces.

The neutrosophic set (NS) is a fresh interpretation of Smarandache's[15, 16] definition of the classical set. The concept of neutrosophic metric spaces was first developed by Kirisci and Simsek [17] and is used to address membership, non-membership, and naturalness. In 2022, Usman[18,19] found out the solution for the nonlinear differential equation for contractive and weakly compatible mappings in neutrosophic metric spaces and discussed the statistically convergent sequence.Some fixed-point findings were demonstrated in the context of neutrosophic metric spaces by Sowndrarajan etal.[20]. Approximate fixed point theorems for weak contractions on neutrosophic normed spaces were proven in 2022 by Jeyaraman [21, 22].

In a neutrosophic n-normed linear space[$\mathcal{N}n\mathcal{N}\mathcal{L}\mathcal{S}$], the concepts of $S_\theta(\Delta)$-convergent and $S_\theta(\Delta)$-Cauchy sequences is presented by establishing a connection between both. We demonstrated the relationship between these ideas and provide their properties using a lacunary density.

2. PRELIMINARIES

Definition 2.1[12]. The following axioms define a continuous t-norm as a binary operation$* : [0,1] \times [0,1] \rightarrow [0,1]$

1. $*$ is continuous, commutative and associative,
2. $\varepsilon * 1 = \varepsilon$ for all $\varepsilon \in [0,1]$,
3. If $\varepsilon \leq \varepsilon'$ and $\theta \leq \theta'$ then $\varepsilon * \theta \leq \varepsilon' * \theta'$, for each $\varepsilon, \varepsilon', \theta, \theta' \in [0,1]$.

Definition 2.2[12]. The following axioms define a continuous t-conorm as a binary operation$\diamond : [0,1] \times [0,1] \rightarrow [0,1]$

1. $\diamond$ is continuous, commutative, and associative,

2. $\varepsilon \diamond 0 = \varepsilon$ for all $\varepsilon \in [0,1]$,
3. If $\varepsilon \leq \theta$ and $\varepsilon' \leq \theta'$ then $\varepsilon \diamond \theta \leq \varepsilon' \diamond \theta'$, for each $\varepsilon, \varepsilon', \theta, \theta' \in [0,1]$.

Definition 2.3. An $\mathcal{NnNLS}$ is the 7-tuple $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$where X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ and $\odot$ continuoust-conorm, μ, ϑ *and* ω are fuzzy sets on$X^n \times (0, \infty)$, μdenotes the degree of membership,ϑ denotes the indeterminacy and ω denotes the non–membership of $(u_1, u_2, \dots, u_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions for every $(u_1, u_2, \dots, u_n) \in X^n$and $s, t > 0$.

a) $0 \leq \mu(u_1, u_2, \dots, u_n, t) \leq 1;\ 0 \leq \vartheta(u_1, u_2, \dots, u_n, t) \leq 1;\ 0 \leq \omega(u_1, u_2, \dots, u_n, t) \leq 1;$
b) $\mu(u_1, u_2, \dots, u_n, t) + \vartheta(u_1, u_2, \dots, u_n, t) + \omega(u_1, u_2, \dots, u_n, t) \leq 3,$
c) $\mu(u_1, u_2, \dots, u_n, t) > 0,$
d) $\mu(u_1, u_2, \dots, u_n, t) = 1$ if and only if $u_1, u_2, \dots, u_n$ are linearly dependent,
e) $\mu(u_1, u_2, \dots, u_n, t)$ is invariant under any permutation of $u_1, u_2, \dots, u_n$,
f) $\mu(u_1, u_2, \dots, \propto u_n, t) = \mu\left(u_1, u_2, \dots, u_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in$ F,
g) $\mu(u_1, u_2, \dots, u_n, s) * \mu(u_1, u_2, \dots, u'_n, t) \leq \mu(u_1, u_2, \dots,, u_n + u'_n, s + t),$
h) $\mu(u_1, u_2, \dots, u_n, t) : (0, \infty) \to [0,1]$ is continuous in t,
i) $\lim_{t\to\infty} \mu(u_1, u_2, \dots, u_n, t) = 1$ *and* $\lim_{t\to 0} \mu(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_n, t) = 0,$
j) $\vartheta(u_1, u_2, \dots, u_n, t) < 1,$
k) $\vartheta(u_1, u_2, \dots, u_n, t) = 0$ if and only if $u_1, u_2, \dots, u_n$ are linearly dependent,
l) $\vartheta(u_1, u_2, \dots, u_n, t)$ is invariant under any permutation of $\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_n$,
m) $\vartheta(u_1, u_2, \dots, \propto u_n, t) = \vartheta\left(u_1, u_2, \dots, u_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in$ F,
n) $\vartheta(u_1, u_2, \dots, u_n, s) \diamond \vartheta(u_1, u_2, \dots, u'_n, t) \geq \vartheta(u_1, u_2, \dots,, u_n + u'_n, s + t),$
o) $\vartheta(u_1, u_2, \dots, u_n, t) : (0, \infty) \to [0,1]$ is continuous in t,
p) $\lim_{t\to\infty} \vartheta(u_1, u_2, \dots, u_n, t) = 0$ *and* $\lim_{t\to 0} \vartheta(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_n, t) = 1,$
q) $\omega(u_1, u_2, \dots, u_n, t) < 1,$
r) $\omega(u_1, u_2, \dots, u_n, t) = 0$ if and only if$u_1, u_2, \dots, u_n$ are linearly dependent,
s) $\omega(u_1, u_2, \dots, u_n, t)$ is invariant under any permutation of $u_1, u_2, \dots, u_n$,

t) $\omega(u_1, u_2, \ldots, \propto u_n, t) = \omega\left(u_1, u_2, \ldots, u_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in F$,

u) $\omega(u_1, u_2, \ldots, u_n, s) \odot \omega(u_1, u_2, \ldots, u'_n, t) \geq \omega(u_1, u_2, \ldots,, u_n + u'_n, s + t)$,

v) $\omega(u_1, u_2, \ldots, u_n, t) : (0, \infty) \rightarrow [0,1]$ is continuous in t,

w) $\lim_{t\rightarrow\infty} \omega(u_1, u_2, \ldots, u_n, t) = 0$ *and* $\lim_{t\rightarrow 0} \omega(u_1, u_2, \ldots, u_n, t) = 1$,

Example 2.1. Let $(X, \|\cdot, \ldots, \cdot\|)$be a linear space with n norms.. Also, let $a * b = ab$,

$a \diamond b = min\{a + b, 1\}anda \odot b = min\{a + b, 1\}$, for every$a, b \in [0,1]$,

$\mu(u_1, u_2, \ldots, u_n, t) = \frac{t}{t+\|u_1,u_2,\ldots,u_n\|}$, $\vartheta(u_1, u_2, \ldots, u_n, t) = \frac{\|u_1,u_2,\ldots,u_n\|}{t+\|u_1,u_2,\ldots,u_n\|}$and

$\omega(u_1, u_2, \ldots, u_n, t) = \frac{\|u_1,u_2,\ldots,u_n\|}{t}$.Then $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is $\mathcal{NnNLS}$.

Definition 2.4. [14] An increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$ is said to be a lacunary sequence[$\mathfrak{LS}$]. Here $I_r = (k_{r-1}, k_r]$are the intervals determined by θ and $q_r = \frac{k_r}{k_{r-1}}$.

Let $K \subseteq \mathbb{N}$. We call the number,$\delta_\theta(K) = \lim_r \frac{1}{h_r}|\{k \in I_r : k \in K\}|$,the$\theta$-density of K, provided the limit exists.

Definition 2.7. [2] Consider a $\mathfrak{LS}$ θ. A sequence $\mathfrak{y} = \{\mathfrak{y}_k\}$ of numbers is said to be lacunary statistically convergent, or, $S_\theta(\Delta)$-convergent, to the number L if for all positive number p, the set $K(p)$ has θ-density zero, where

$K(p) = \{k \in \mathbb{N}: |\mathfrak{y}_k - L| \geq p\}$ and this is written as $S_\theta - \lim_{k\rightarrow\infty} \mathfrak{y}_k = L$.

3. Δ-CONVERGENCE AND LACUNARY Δ-STATISTICAL CONVERGENCE IN $\mathcal{NnNLS}$

Definition 3.1. Assume that $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$. A sequence $x = \{x_k\} \in X$ is called Δ-Convergent to $L \in X$ in relation to the Neutrosophic n-norm $(\mu, \vartheta, \omega)^n$ if, for all$\varepsilon > 0, t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon$, $\vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$ and

$\omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$, for all $k \geq k_0$, where $k \in \mathbb{N}$ and $\Delta x_k = (x_k - x_{k-1})$. It's indicated by $(\mu, \vartheta, \omega)^n$-$\lim_{k\to\infty} \Delta x_k = L$ or $\Delta x_k \to L$ as $k \to \infty$.

Definition 3.2. Assume that$(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$. A sequence $x = \{x_k\} \in X$ is called lacunary Δ-statistical convergent or $S_\theta(\Delta)$-convergent to $L \in X$ in relation to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$assuming that for each $\varepsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$,

$$\delta_\theta(\Delta)\left(\left\{\begin{array}{c} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \\ or \quad \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{array}\right\}\right) = 0,$$

or equivalently $\delta_\theta(\Delta)\left(\left\{\begin{array}{r} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon, \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon \end{array}\right\}\right)$ $= 1$.

It's indicated by$S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$or $x_k \to L(S_\theta(\Delta))$.

We can get the lemma by using definition (3.2) and the properties of the θ-density.

Lemma 3.1. Assume that$(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$ andθ be a $\mathfrak{LS}$. Next, for each$\varepsilon > 0, t > 0$and $y_1, y_2, \ldots, y_{n-1} \in X$,the following claims are interchangeable:

i. $S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$,

ii. $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon\})$

$$= \delta_\theta(\Delta)(\{k \in \mathbb{N}: \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\})$$

$$= \delta_\theta(\Delta)(\{k \in \mathbb{N}: \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}) = 0,$$

iii. $\delta_\theta(\Delta)\left(\left\{\begin{array}{r} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon, \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon \end{array}\right\}\right) = 1$,

iv. $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon\})$

$$=\delta_\theta(\Delta)(\{k \in \mathbb{N}: \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon\})$$

$$=\delta_\theta(\Delta)(\{k \in \mathbb{N}: \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon\}) = 1,$$

v. $S_\theta - \lim \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) = 1,$

$S_\theta - \lim \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) = 0$and

$S_\theta - \lim \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) = 0.$

Theorem 3.1. Let $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ be a $\mathcal{NnNLS}$ and θ be a $\mathfrak{LS}$. If $(\mu, \vartheta, \omega)^n - \lim \Delta x = L$, then $S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$.

Proof: Let $(\mu, \vartheta, \omega)^n - \lim \Delta x = L$. Then, for each$\varepsilon > 0, t > 0$and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon,$

$\vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$and$\omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) < \varepsilon,$ for all $k \geq k_0$.

Hence the set

$$\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{matrix}\right\}$$ has a set of limited terms.

Since the lacunary density of each finite subset of ℕ is 0,

$$\delta_\theta(\Delta)\left(\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{matrix}\right\}\right) = 0,$$

that is, $S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L.$

Theorem (3.1)'s converse is not generally true, as is seen from the case that follows.

Example 3.1. Consider$X = \mathbb{R}^n$with $\|x_1, x_2, \ldots, x_n\| =$
$$abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1,2, \ldots, n$ and let $a * b = ab,$
$a \diamond b = \min\{a + b, 1\}, a \odot b = \min\{a + b, 1\}$, for all $a, b \in [0,1]$. Now, for all $y_1, y_2, \ldots, y_{n-1}, x \in \mathbb{R}^n$ and $t > 0, \mu(y_1, y_2, \ldots, y_{n-1}, x, t) = \frac{t}{t + \|y_1, y_2, \ldots, y_{n-1}, x\|},$

$\vartheta(y_1, y_2, \dots, y_{n-1}, x, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, x\|}{t + \|y_1, y_2, \dots, y_{n-1}, x\|}$and

$\omega(y_1, y_2, \dots, y_{n-1}, x, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, x\|}{t}$. Then $(\mathbb{R}^n, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$.

Let $I_r$and $h_r$have the meanings they have in definition (2.6).

Define a sequence with terms provided by $x = \{x_k\}$.

$$x_k = \begin{cases} \left(\frac{(n - [\sqrt{h_r}] + 1)(-n + [\sqrt{h_r}])}{2}, 0, \dots, 0\right) \in \mathbb{R}^n \text{ if } 1 \le k \le n - [\sqrt{h_r}], \\ \left(-\frac{1}{2}k^2 + \frac{1}{2}k, 0, \dots, 0\right) \in \mathbb{R}^n \text{ if } n - [\sqrt{h_r}] + 1 \le k \le n, \\ \left(-\frac{1}{2}n^2 + \frac{1}{2}n, 0, \dots, 0\right) \in \mathbb{R}^n \text{ if } k > n \end{cases}$$ such that

$$\Delta x_k = \begin{cases} (k, 0, \dots, 0) \in \mathbb{N} & \text{if } n - [\sqrt{h_r}] + 1 \le k \le n, \\ (0, 0, \dots 0) \in \mathbb{N} & \text{otherwise.} \end{cases}$$

For every $0 < \varepsilon < 1$ and for any $y_1, y_2, \dots, y_{n-1} \in X, t > 0$, let

$$K(\varepsilon, t) = \left\{ \begin{array}{c} k \in I_r : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \le 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \ge \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \ge \varepsilon\,. \end{array} \right\}.$$

Now, $K(\varepsilon, t) = \left\{ k \in I_r : \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\| \ge \frac{\varepsilon t}{1 - \varepsilon} > 0 \right\}$

$$\subseteq \{k \in I_r : \Delta x_k = (k, 0, \dots, 0) \in \mathbb{R}^n\}.$$

Thus, we have$\frac{1}{h_r} |\{k \in I_r : k \in K(\varepsilon, t)\}| \le \frac{[\sqrt{h_r}]}{h_r} \to 0$ as $r \to \infty$.

Hence,$S_\theta^{(\mu, \vartheta, \omega)^n}(\Delta) - \lim x = 0$. As opposed to that, $x = \{x_k\}$ in X is not Δ-convergent to 0in relation to the neutrosophic n-norm, since

$$\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k, t) = \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}$$

$$= \left\{ \begin{array}{ll} \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|} & \text{if } n - [\sqrt{h_r}] + 1 \le k \le n, \\ 1 & \text{otherwise.} \end{array} \right\} \le 1,$$

$$\vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}$$

$$= \begin{Bmatrix} \frac{\|y_1,y_2,\dots,y_{n-1},\Delta x_k\|}{t+\|y_1,y_2,\dots,y_{n-1},\Delta x_k\|} & \text{if } n - \left[\sqrt{h_r}\right] + 1 \le k \le n, \\ 0 & \text{otherwise.} \end{Bmatrix} \ge 0 \text{and}$$

$$\omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}{t}$$

$$= \begin{Bmatrix} \frac{\|y_1,y_2,\dots,y_{n-1},\Delta x_k\|}{t} & \text{if } n - \left[\sqrt{h_r}\right] + 1 \le k \le n, \\ 0 & \text{otherwise.} \end{Bmatrix} \ge$$

0.

The theorem's proof is now complete.

Theorem 3.2. Let $(X, \mu, \vartheta, \omega, *, \diamondsuit, \odot)$ be a $\mathcal{N}n\mathcal{N}\mathcal{L}\mathcal{S}$. Then $S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$, if andonly if there exists an ascending sequence $K = \{k_n\}$occurs of the natural numbers like that $S_\delta(\Delta)(K) = 1$ and $(\mu, \vartheta, \omega)^n - \lim_{k \in K} \Delta x_k = L$.

Proof: Necessity. Assume that $S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$. Then, for every $y_1, y_2, \dots, y_{n-1} \in X$, $t > 0$ and $j = 1,2, \dots$

$$K(j,t) = \begin{Bmatrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \text{and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \end{Bmatrix} \text{and}$$

$$M(j,t) = \begin{Bmatrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \le 1 - \frac{1}{j} \text{or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \ge \frac{1}{j} \text{and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \ge \frac{1}{j}\ . \end{Bmatrix}.$$

Then $\delta_\theta(\Delta)(M(j,t)) = 0$ since,

$$K(j,t) \supset K(j+1,t) \tag{3.1}$$

and

$$\delta_\theta(\Delta)(K(j,t)) = 1, \tag{3.2}$$

for $t > 0$ and $j = 1,2,...$. Now, we must demonstrate that to $k \in K(j,t)$ imagine that in some cases $k \in K(j,t)$, $x = \{x_k\}$ not Δ-convergent to L in relation to neutrosophic n-norm $(\mu,\vartheta,\omega)^n$. Consequently, there is $\alpha > 0$ and a non-negative integer k_0 like that $\mu(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \leq 1 - \alpha$ or $\vartheta(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \geq \alpha$ and $(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \geq \alpha$, for all $k \geq k_0$.

Let $\alpha > \frac{1}{j}$ and

$$K(j,t) = \left\{ \begin{array}{r} k \in \mathbb{N}: \mu(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) > 1 - \alpha, \\ \vartheta(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) < \alpha \text{ and} \\ \omega(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) < \alpha. \end{array} \right\}.$$

Then $\delta_\theta(\Delta)(K(\alpha,t)) = 0$. Since $\alpha > \frac{1}{j}$ by (**3.1**) we have $\delta_\theta(\Delta)\big(K(j,t)\big) = 0$, which contradicts by **Eq.** (**3.2**).

Sufficiency. Let's say there is an ascending series $K = \{k_n\}$ of the natural numbers like that $\delta_\theta(\Delta)(K) = 1$ and $(\mu,\vartheta,\omega)^n - \lim_{k \in K} \Delta x_k = L$,

i.e., for every $y_1,y_2,...,y_{n-1} \in X$, $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) > 1 - \varepsilon$, $\vartheta(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) < \varepsilon$ and $\omega(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) < \varepsilon$.

$$\text{Let } M(\varepsilon,t) = \left\{ \begin{array}{c} k \in \mathbb{N}: \mu(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \geq \varepsilon \text{ and} \\ \omega(y_1,y_2,...,y_{n-1},\Delta x_k - L,t) \geq \varepsilon \end{array} \right\}$$

$\subseteq \left\{k_{n_0+1}, k_{n_0+2}, ...\right\}$ and consequently $\delta_\theta(\Delta)\big(M(\varepsilon,t)\big) \leq 1 - 1 = 0$.

Hence ${S_\theta}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$. The theorem's proof is now complete.

Theorem 3.3. Let $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ be a $\mathcal{N}n\mathcal{NLS}$. Then ${S_\theta}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$, if and only if a convergent sequence $y = \{y_k\}$ exists and a lacunary Δ-statistically null sequence $z = \{z_k\}$ in relation to the neutrosophic n-norm $(\mu,\vartheta,\omega)^n$ like that $(\mu,\vartheta,\omega)^n - \lim \mathrm{y} = \mathrm{L}$,

$\Delta\mathrm{x} = \mathrm{y} + \Delta\mathrm{z}$ and $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \Delta z_k = 0\}) = 1$.

Proof: Necessity. Imagine that ${S_\theta}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$ and

$$K(j,t) = \left\{ \begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j}\ . \end{matrix} \right\}.$$

Using theorem (3.2) for any $y_1, y_2, \dots, y_{n-1} \in X, t > 0$and $j \in \mathbb{N}$, we can construct a natural number increasing index sequence $\{r_j\}$ such that $r_j \in K(j,t), \delta_\theta(\Delta)\big(K(j,t)\big) = 1$ and thus, we draw the conclusion that for each $r > r_j$ $(j \in \mathbb{N})$,

$$\frac{1}{h_r} \left\{ \begin{matrix} k \in I_r: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j}\ . \end{matrix} \right\} > \frac{j-1}{j}.$$

Define $y = \{y_k\}$ and $z = \{z_k\}$ as follows. If $1 < k < r_1$, we set $y_k = \Delta x_k$ and $z_k = 0$.

Now suppose that $j \geq 1$and$r_j < k \leq r_{j+1}$.

If $k \in K(j,t)$, i.e., $\mu(y_1, y_2, \dots, y_{n-1},\ \ \Delta x_k - L, t) > 1 - \frac{1}{j}$,

$\vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j}$and $\omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j}$,

we set $y_k = \Delta x_k$ and $\Delta z_k = 0$.

Otherwise,$y_k = L$and $\Delta z_k = \Delta x_k - L$. Thus, it is evident that$\Delta x = y + \Delta z$.

We assert that$(\mu, \vartheta, \omega)^n - \lim y = L$. Let $\varepsilon > \frac{1}{j}$. If $k \in K(j,t)$ for all $k > r_j$,

$\mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) > 1 - \varepsilon, \vartheta(y_1, y_2, \dots, y_{n-1}, y_k - L, t) < \varepsilon$ and $\omega(y_1, y_2, \dots, y_{n-1}, y_k - L, t) < \varepsilon$.

Because ε was arbitrary, we have established the assertion. Next, assert that $z = \{z_k\}$ is a lacunary Δ-statistically null sequence with regard to the neutrosophic n-norm$(\mu, \vartheta, \omega)^n$,

i.e.,$S_\theta{}^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim z = 0$.

It suffices to see that $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \Delta z_k = 0\}) = 1$to support the assertion.This results from the fact that:

$$|\{k \in I_r: \Delta z_k = 0\}| \leq \left|\left\{\begin{matrix} k \in I_r: \mu(y_1, y_2, \dots, y_{n-1}, \Delta z_k - L, t) > 1 - \varepsilon, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta z_k - L, t) < \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta z_k - L, t) < \varepsilon \end{matrix}\right\}\right|,$$

for each$r \in \mathbb{N}$ and $\varepsilon > 0$. We demonstrate if $\delta > 0$ and $j \in \mathbb{N}$ such that $\frac{1}{j} < \delta$, then

$\frac{1}{h_r}|\{k \in I_r: \Delta z_k = 0\}| > 1 - \delta$for all $r > r_j$. Remember from the design that if $k \in K(j,t)$, then $\Delta z_k = 0$ for $r_j < k \leq r_{j+1}$. Now, for $t > 0$and $s \in \mathbb{N}$, let

$$K(s,t) = \left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \text{and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s}\ . \end{matrix}\right\}.$$

For $s > j$ and $r_s < k \leq r_{s+1}$ by (3.6.2),

$$K(s,t) = \left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \text{and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \end{matrix}\right\} \subset \{k \in \mathbb{N}: \Delta z_k = 0\}.$$

Consequently, if $r_s < k \leq r_{s+1}$and $s > j$, then

$$\frac{1}{h_r}|\{k \in I_r: \Delta z_k = 0\}| \geq \frac{1}{h_r}\left|\left\{\begin{matrix} k \in I_r: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s}, \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \text{and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \end{matrix}\right\}\right| > 1 - \frac{1}{s} > 1 - \frac{1}{j} > 1 - \delta.$$

Thus, we obtain $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \Delta z_k = 0\}) = 1$, which supports the assertion.

Sufficiency Let x, y and z be sequence such that $(\mu, \vartheta, \omega)^n - \lim y = L$, $\Delta x = y + \Delta z$ and $\delta_\theta(\Delta)(\{k \in \mathbb{N}: \Delta z_k = 0\}) = 1$.

Then for each $y_1, y_2, \dots, y_{n-1} \in X, \varepsilon > 0$ and $t > 0$, we have

$$\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{matrix}\right\}$$

$$\subseteq \left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \end{matrix}\right\} \cup \{k \in \mathbb{N}: \Delta z_k \neq 0\}.$$

Therefore, $\delta_\theta(\Delta)\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{matrix}\right\}$

$$\leq \delta_\theta\left(\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \end{matrix}\right\}\right) + \delta_\theta(\Delta)\{k \in \mathbb{N}: \Delta z_k \neq 0\}.$$

Since $(\mu, \vartheta, \omega)^n - \lim y = L$, the set

$$\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \end{matrix}\right\}$$

includes only a finite number of terms,

thus $\delta_\theta\left(\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \end{matrix}\right\}\right)$.

Additionally, according to supposition $\delta_\theta(\Delta)\{k \in \mathbb{N}: \Delta z_k \neq 0\}$. Hence,

$$\delta_\theta(\Delta)\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \end{matrix}\right\} = 0 \text{and}$$

Consequently, $S_\theta^{(\mu,\vartheta,\omega)^n}(\Delta) - \lim x = L$.

4. SEQUENCES IN $\mathcal{NnNLS}$ WITH LACUNARY Δ-STATISTICAL CAUCHY STRUCTURE

Definition 4.1. Assume that $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$. A sequence $x = \{x_k\} \in X$ is called Δ-cauchy in relation to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$ if, for each $\varepsilon > 0, t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in \mathbb{X}$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) > 1 - \varepsilon$, $\vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) < \varepsilon$ and $\omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) < \varepsilon$ for all $k, m \geq k_0$.

Definition 4.2. Assume that $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$. A sequence $x = \{x_k\} \in X$ is called lacunary Δ-statistically cauchy or $S_\theta(\Delta)$-cauchy in relation to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$ if, for each $\varepsilon > 0, t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists a number $m \in \mathbb{N}$ satisfying

$$\delta_\theta(\Delta)\left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_m, t) \geq \varepsilon \end{matrix}\right\} = 0.$$

Theorem 4.1. Consider $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ be a $\mathcal{NnNLS}$. If a sequence $x = \{x_k\} \in X$ is lacunary Δ-statistically convergent in relation to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$ if and only if it is lacunary Δ-statistically cauchy in relation to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$.

Proof: Let L be the convergence point of the lacunary Δ-statistically convergent sequence $x = \{x_k\}$. Choose > 0, for a given $\varepsilon > 0$ such that $(1 - \varepsilon) * (1 - \varepsilon) > 1 - s$, $\varepsilon \diamond \varepsilon < s$ and $\varepsilon \odot \varepsilon < s$.

$$\text{Let } A(\varepsilon, t) = \left\{\begin{matrix} k \in \mathbb{N}: \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) \leq 1 - \varepsilon \text{ or} \\ \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) \geq \varepsilon \text{ and} \\ \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) \geq \varepsilon \end{matrix}\right\}.$$

Then for any $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$,

$$\delta_\theta(\Delta)\big(A(\varepsilon, t)\big) = 0, \tag{4.1}$$

it suggests that $\delta_\theta(\Delta)\big(A^c(\varepsilon, t)\big) = 1$. Let $q \in A^c(\varepsilon, t)$.

Consequently, $\mu\big(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\big) > 1 - \varepsilon$,

$\vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon$ and $\omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon$.

Now, Let

$$B(s,t) = \left\{ \begin{matrix} k \in \mathbb{N}: \mu\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t/2\right) \leq 1 - s \text{ or} \\ \vartheta\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t/2\right) \geq s \text{ and} \\ \omega\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t/2\right) \geq s \end{matrix} \right\}.$$

We must demonstrate that$B(s,t) \subset A(\varepsilon, t)$. Let $k \in B(s,t) \cap A^c(\varepsilon, t)$.

Hence $\mu\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right) \leq 1 - s$,

$\mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) > 1 - \varepsilon$,

in particular,$\mu\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right) > 1 - \varepsilon$.

Then$1 - s \geq \mu\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right)$

$\geq \mu(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) * \mu\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right)$

$> (1 - \varepsilon) * (1 - \varepsilon) > 1 - s$, which is not possible.

On the other hand, $\vartheta\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right) \geq s$ and

$\vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon$, in particular,

$\vartheta\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right) < \varepsilon$.

Hence, $s \leq \vartheta\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right)$

$\leq \vartheta(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) \diamondsuit \vartheta\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right)$

$< \varepsilon \diamondsuit \varepsilon < s$, which is not possible.

Similarly, $\omega\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right) \geq s$, $\omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon$,

in particular, $\omega\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right) < \varepsilon$.

Hence,$s \leq \omega\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - \Delta x_q, t\right)$

$\leq \omega(y_1, y_2, \ldots, y_{n-1}, \Delta x_k - L, t/2) \odot \omega\left(y_1, y_2, \ldots, y_{n-1}, \Delta x_q - L, t/2\right)$

$< \varepsilon \odot \varepsilon < s$, which is not possible.

Hence, $B(s,t) \subset A(\varepsilon,t)$ and by **Eq. (4.1)** $\delta_\theta(\Delta)\big(B(\varepsilon,t)\big) = 0$.

This demonstrates that, with respect to the Neutrosophic n-norm $(\mu,\vartheta,\omega)^n$, x is lacunaryΔ-statistically cauchy.

Let $x = \{x_k\}$, on the other hand, be lacunary Δ-statisticallycauchy but not Δ-statistically converging with regard to the neutrosophic n-norm $(\mu,\vartheta,\omega)^n$.

Choose > 0, for a given $\varepsilon > 0$such that$(1-\varepsilon) * (1-\varepsilon) > 1-s$, $\varepsilon \diamond \varepsilon < s$ and $\varepsilon \odot \varepsilon < s$.$x$ is not lacunary Δ-convergent, thus.

$$\mu(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_m, t)$$
$$\geq \mu(y_1,y_2,\dots,y_{n-1},\Delta x_k - L, t/2) * \mu\big(y_1,y_2,\dots,y_{n-1},\Delta x_q - L, t/2\big)$$
$$> (1-\varepsilon) * (1-\varepsilon) > 1-s,$$
$$\vartheta(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_m, t)$$
$$\leq \vartheta(y_1,y_2,\dots,y_{n-1},\Delta x_k - L, t/2) \diamond \vartheta\big(y_1,y_2,\dots,y_{n-1},\Delta x_q - L, t/2\big)$$
$$< \varepsilon \diamond \varepsilon < s$$
$$\omega(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_m, t)$$
$$\leq \omega(y_1,y_2,\dots,y_{n-1},\Delta x_k - L, t/2) \odot \omega\big(y_1,y_2,\dots,y_{n-1},\Delta x_q - L, t/2\big)$$
$$< \varepsilon \odot \varepsilon < s.$$

Therefore $\delta_\theta(\Delta)\big(E^c(s,t)\big) = 0$, where

$$B(s,t) = \left\{\begin{matrix} k \in \mathbb{N}: \mu\big(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_q, t\big) \leq 1-s \text{ or} \\ \vartheta\big(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_q, t\big) \geq s \text{ and} \\ \omega\big(y_1,y_2,\dots,y_{n-1},\Delta x_k - \Delta x_q, t\big) \geq s \end{matrix}\right\} \text{and so}$$

$\delta_\theta(\Delta)\big(E(s,t)\big) = 1$.It is in conflict with the fact that x was lacunary Δ-statistically cauchy with respect to the neutrosophic n-norm $(\mu,\vartheta,\omega)^n$. In light of the neutrosophic n-norm $(\mu,\vartheta,\omega)^n$, x must be lacunary Δ-statistically convergent.

Corollary 4.1. Let $(X,\mu,\vartheta,\omega,*,\diamond,\odot)$ be a $\mathcal{N}n\mathcal{NLS}$ and θ be a $\mathfrak{LS}$. Then, for any sequence $x = \{x_k\} \in X$, the following circumstances are comparable:

i. With respect to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$, x is $S_\theta(\Delta)$-Convergent.
ii. With respect to the neutrosophic n-norm $(\mu, \vartheta, \omega)^n$, x is $S_\theta(\Delta)$-Cauchy.
iii. There is a natural number sequence of increasing length $K = \{k_n\}$ such that $\delta_\theta(\Delta)(K) = 1$and the subsequence $\{x_{k_n}\}$ is $\delta_\theta(\Delta)$-Cauchy with regard to the Neutrosophic n-norm $(\mu, \vartheta, \omega)^n$.

CONCLUSION

The current study has introduced the developments of lacunary, established various properties of $S_\theta(\Delta)$-convergent and $S_\theta(\Delta)$- cauchy sequences in $\mathcal{NnNLS}$. Furthermore, this concept can be used in the future to find the fixed point results of $\mathcal{NnNLS}$ for various types of mapping by using various contraction conditions.

Conflict of Interest

There is no conflict of interest among author of this article.

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