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A Curious Property of Octagons

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Abstract

The famous theorem of Van Aubel for quadrilaterals postulates that if squares are built externally on the sides of any quadrilateral, then the two segments that join the opposing centers of these squares are congruent and orthogonal. Inspired by this result and also by the results of Krishna, in this article we will prove the following result of plane geometry: each octagon is associated with a parallelogram, in some cases the parallelogram in question can be degenerate at a point or a segment. This is possible because of complex numbers and basics of analytical geometry.

Keywords: octagon, plane geometry, Van Aubel theorem

1. Introduction

Krishna [1, 2], Oai [3] and Silva [4] has recently proved interesting properties of octagons which bring forth the fact that Euclidean plane geometry is not closed, as one might think, without it being possible to discover other relations not yet observed by the scientific community. These works are similar to Van Aubel's result regarding the quadrilaterals, as we can see in Nishiyama [5].

Taking as motivation the work of Krishna [1] and Santos [6], this article concludes a result in plane geometry that concerns parallelograms associated with octagons. The demonstration involves high school concepts such as analytical geometry. This article can be useful in providing the teachers of basic or higher education with students who want to delve into the subject through activities parallel to regular teaching.

2. Motivational Problems

This section depicts the results that motivated the discovery of the geometrical property to be demonstrated in this article.

According to Krishna [1], given an octagon $A_1A_2 \dots A_8$ (convex or not) and equilateral triangles $A_iA_{i+1}B_i$ i module 8, built all internally or all externally on the sides of the octagon, and given that C_i is the midpoint of B_iB_{i+1} then, if P_1 is the midpoint of C_1C_5, P_2 is the midpoint of C_2C_6, P_3 is the midpoint of C_3C_7 and P_4 is the midpoint of C_4C_8, it is concluded that $P_1P_2P_3P_4$ is a parallelogram as we can see in figure 1. Moreover, the centers of the parallelograms resulting from the internal and external constructions remain the same.

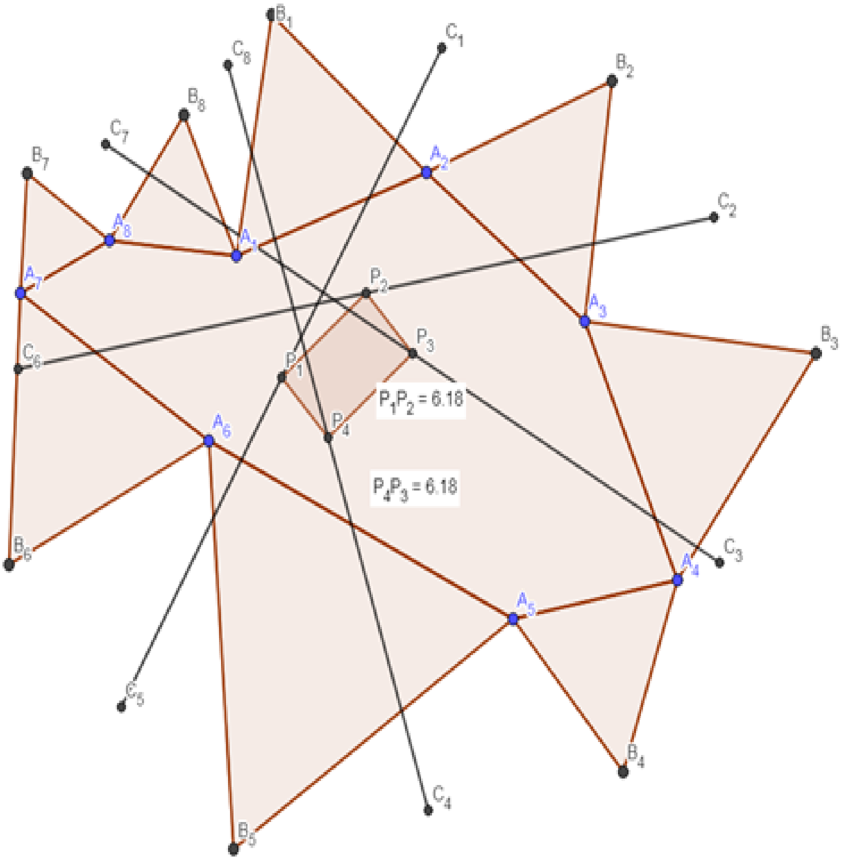

Figure 1. Triangles built on the sides of any octagon

This result has an interesting analog in relation to dodecagons. In fact, according to Krishna [1], given a dodecagon (convex or not) $A_1A_2 \ldots A_{12}$ and equilateral triangles $A_iA_{i+1}B_i$, i module 12, built all internally or all externally on the sides of the dodecagon as shown in figure 2, and given that C_i is the midpoint of B_iB_{i+1} then, if P_i is the midpoint of C_iC_{i+6} it turns out that

1) P_1P_4, P_2P_5 and $P_3\ P_6$ are congruent.
2) If Q_1, Q_2 and Q_3 are the intersections of the segments P_1P_4, P_2P_5 and P_3P_6, then $Q_1Q_2Q_3$ is an equilateral triangle.

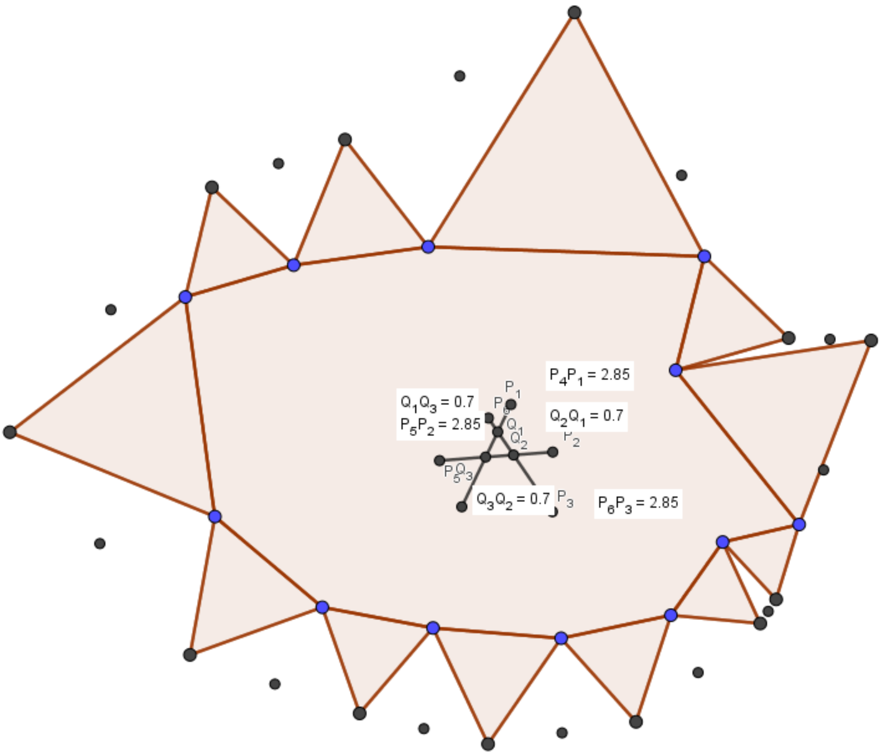

Figure 2. Triangles built on the sides of a dodecagon

Another result from the same author posits that given an octagon (convex or not) $A_1A_2 \dots A_8$ and squares $B_1A_1A_2B_2$, $B_3A_2A_3B_4, \dots, B_{15}A_8A_1B_{16}$, built all internally or all externally on the sides of the octagon, and given that $C_1, \dots, C_8$ are the midpoints of B_2B_3, $B_4B_5, B_6B_7, \dots, B_{16}B_1$ then, if P_1 is the midpoint of C_1C_5, P_2 is the midpoint of C_2C_6, P_3 is the midpoint of C_3C_7 and P_4 is the midpoint of C_4C_8, it is concluded that $P_1P_2P_3P_4$ is an iso-ortho-diagonal quadrilateral, that is, its diagonals are congruent and orthogonal as we can see in figure 3.

We are used to studying the properties of polygons that are regular, such as pentagons, hexagons, heptagons or regular octagons. We do not expect any octagon, regular or otherwise, to have interesting properties.

In this article, we will prove a curious property of octagons; its demonstration can be presented to high school students because it uses the basics of analytical geometry. This property can also be illustrated and verified with students through GeoGebra software, which can become a rich experience of research and mathematical curiosity. The result is a generalization of the work of Krishna [1] illustrated in figure 1 of this article and also a generalization of the work of Santos [6].

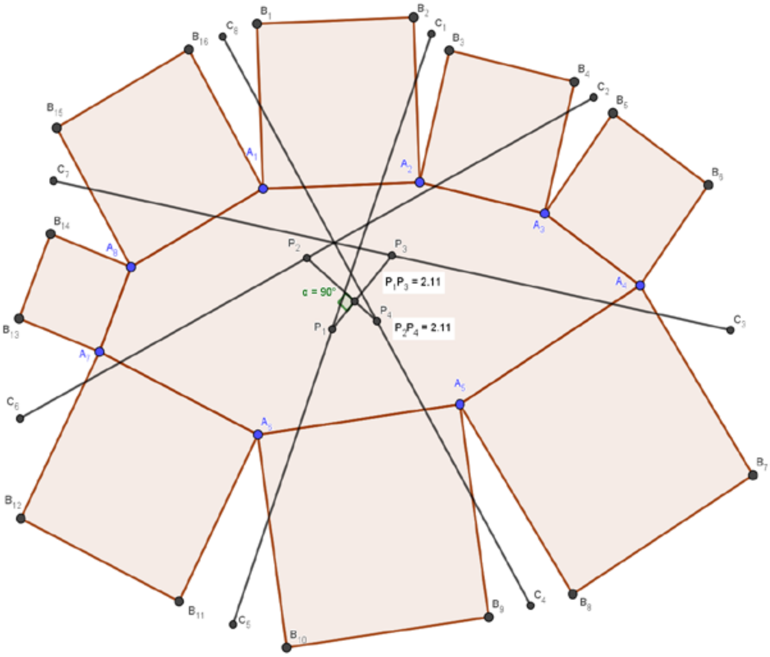

Figure 3. Squares built on the sides of an octagon

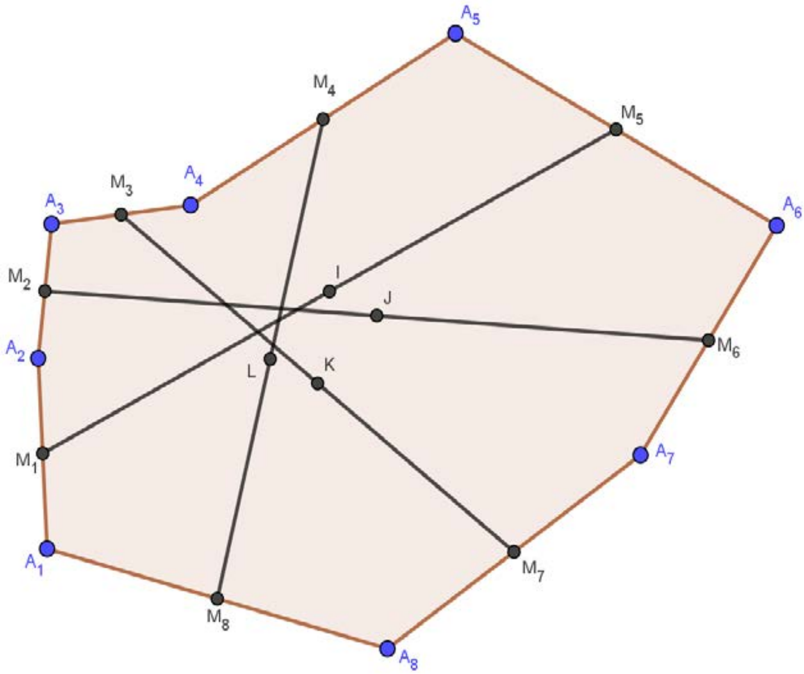

Figure 4. Parallelogram associated with a general octagon

3. A Property of the Octagons

Consider a general octagon $A_1A_2 \dots A_8A_1$ (concave or convex). Considering $M_1,\ M_2, \dots$ and M_8 which are the midpoints of the sides $A_1A_2,\ A_2A_3, \dots, A_8A_1$, respectively, let I, J, K and L be the midpoints of the segments $M_1M_5,\ M_2M_6,\ M_3M_7$ e M_4M_8, respectively (I ask the more attentive reader to forgive me the use of the letter I to name a point other than the one in the center of a triangle).

In these conditions, $\overline{IJ} = \overline{LK}$ and $\overline{JK} = \overline{IL}$. Consequently, if $IJKL$ forms a non-degenerate quadrilateral, then $IJKL$ is a parallelogram, that is, it has parallel opposite sides (remember that a quadrilateral is a parallelogram if and only if the opposite sides are congruent).

In order to prove this property, consider each point in figure 4 as a point in the Cartesian plane. Therefore, by defining the midpoint of a segment in the plan, we have

$$M_1 = \frac{A_1 + A_2}{2}, M_2 = \frac{A_2 + A_3}{2}, \dots, M_8 = \frac{A_8 + A_1}{2}.$$

Thus

$$I = \frac{M_1 + M_5}{2} = \frac{A_1 + A_2 + A_5 + A_6}{4},$$
$$J = \frac{M_2 + M_6}{2} = \frac{A_2 + A_3 + A_6 + A_7}{4},$$
$$K = \frac{M_3 + M_7}{2} = \frac{A_3 + A_4 + A_7 + A_8}{4} \text{ and}$$
$$L = \frac{M_4 + M_8}{2} = \frac{A_4 + A_5 + A_8 + A_1}{4}.$$

The length of a segment AB in the Cartesian plane is given by $|B - A|$. Therefore, the length of IJ is given by $|J - I| =$

$$\left|\frac{A_2 + A_3 + A_6 + A_7}{4} - \frac{A_1 + A_2 + A_5 + A_6}{4}\right| = \left|\frac{A_3 + A_7 - A_1 - A_5}{4}\right|.$$

The length of LK, in turn, is given by $|K - L| =$

$$\left|\frac{A_3 + A_4 + A_7 + A_8}{4} - \frac{A_4 + A_5 + A_8 + A_1}{4}\right| = \left|\frac{A_3 + A_7 - A_5 - A_1}{4}\right| = |J - I|.$$

That is, the segments IJ e LK are congruent. Analogously, it is shown that JK e IL are congruent which concludes the proof of this interesting property of octagons.

4. Concluding Remarks

It is noted that in case of a regular octagon the points I, J, K e L coincide, so it can be concluded that $\overline{IJ} = \overline{LK} = \overline{JK} = \overline{IL} = 0$. In non-regular octagons, in cases where such points form a quadrilateral, it is therefore proven that they form a parallelogram, simultaneously.

A pertinent question remains that if it is possible to obtain the proof of this result by means of positional geometry, perhaps by means of the medium base theorem and without the use of analytical geometry? We haven't put it to test yet but the challenge has been set.

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