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Extension of Some Common Fixed-Point Theorems in Neutrosophic Metric Spaces Via Control Function

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ABSTRACT

The aim of this paper is to prove some important fixed-point theorems in the context of the neutrosophic metric space, which is a generalization of the fuzzy Banach fixed-point theorem, by utilizing the control function. Also, certain fixed-point theorems in the G-complete neutrosophic metric space are proved and discussed by utilizing the alternating distance function (ADF) and defined neutrosophic (Φ, Ψ)*-weak contraction. The current study supports the results with some non-trivial examples. Furthermore, it also supports the main result with an application of the Fredholm integral equation.*

Keywords: fixed-point, control function, neutrosophic metric space, (Φ, Ψ)-weak contraction

INTRODUCTION

The idea of fuzzy sets (FSs) was first suggested by Zadeh [1] in 1965. Since then, a number of researchers have thoroughly studied the theory of FSs along with its applications in order to use this notion in topology and analysis. In a similar effort, Atanassov [2] developed the notion of intuitionistic fuzzy sets (IFSs) in 1986 as an extension of FSs. Alaca et al. [3] established the notion of intuitionistic fuzzy metric space (IFMSs) based on the concept of IFSs. As an extension of fuzzy metric space (FMS) in the sense of Kramosil and Michalek [4], Park [5] extended the idea of IFMSs.

In IFMSs, Turkoglu et al. [6] established the compatible maps of types (α) and (β) and established some relations between them.

Khan et al. [7] used the concept of modifying distance in metric fixed-point findings in 1984. In order to cope with the relatively novel kinds of fixed-point issues, both the altering and control functions are used to change the metric distance between two points. When changing distance is involved, the application of specific methods may be necessary since the triangle of inequality is not always immediately relevant. A common fixed-point theorem for weakly compatible maps in IFMSs was established by Sanjay et al. [8]. Also, they discussed some results related to the variants of R-weakly commuting mappings.

Saleem et al. [9] introduced two new classes of mappings known as λ-enriched, strictly pseudocontractive, mappings and ΦT-enriched Lipshitizian mappings in the setup of a real Banach space. Saleem et al. [10] introduced the notion of intuitionistic extended fuzzy b-metric-like spaces and established some fixed-point theorems in their setting. Ali et al. [11] proved various unique fixed-point results for contractive and weakly compatible mappings in the sense of neutrosophic metric spaces (NMSs). Omeri et al. [12, 13] obtained common fixed-point theorems in the neutrosophic cone metric space. Also, the notion of -weak contraction was defined by them in the neutrosophic cone metric space by using the idea of alternating distance function (ADF).

Hussain et al. [14] introduced the notions of pentagonal controlled fuzzy metric space and fuzzy controlled hexagonal metric space as generalizations of fuzzy triple controlled metric spaces and fuzzy extended hexagonal b-metric spaces. They also used a control function in fuzzy controlled hexagonal metric space and introduced five non-comparable control functions in pentagonal controlled fuzzy metric spaces. Jhangeer et al. [15] derived Atangana–Baleanu derivative in Riemann–Liouville sense is practiced in the fractional modeling of the weakly non-linear shallow water wave equation. Asjid et al. [16] obtained new traveling wave solutions of the double dispersive equation with a more general mathematical technique known as the direct algebraic extended method. Zhang et al. [17] extended the direct algebraic technique to attain the new exact solitary wave solutions of the nonlinear fractional couple Drinfeld–Sokolov–Wilson system. Asjad et al. [18] investigated the Hirota equation which has a significant role in applied sciences including maritime, coastal engineering, ocean sciences, and the main sources of environmental action due to energy transportation on floating anatomical structures.

In 2014, Beg et al. [19] established the notion of (Φ, ψ)-weak contraction in the context of IFMSs and derived several FP results by utilizing the ADF. Kirişci and Simsek [20] introduced the neutrosophic metric spaces (NMSs) technique, which addresses membership, non-membership, and naturalness functions. Sowndrarajan et al. [21] presented some FP results for NMSs. For more related results, see [22-24].

In this manuscript, the following contractive condition has been generalized:

$$\frac{1}{\mathcal{M}(\zeta(\varpi),\zeta(\omega),\tau)}-1 \leq \Upsilon\left(\frac{1}{\mathcal{M}(\varpi,\omega,\tau)}-1\right).$$

Moreover, certain common fixed-point (CFP) theorems have been established in the context of G-complete NMS using ADF. Additionally, CFP results have been established in the context of NMS for functions that meet a certain inequality involving three control functions (CFs).

Definition 1.1. [22] A binary operation $*$ on $[0, 1]$ is said to be continuous t–norm (CTN) if the following circumstances are fulfilled:

(T1) $*$ is continuous, commutative, and associative;
(T2) $\theta * 1 = \forall \theta \in [0,1]$;
(T3) $\theta * \partial \leq c * d$, whenever $\theta \leq c$ and $\partial \leq d$, $\forall \in \theta, \partial, c, d \in [0,1]$.

Definition 1.2. [24] A binary operation $\lozenge$ on [0, 1] is said to be continuous t–conorm (CTCN) if (T1)-(T3) and the following condition is fulfilled:

(T4) $\theta \lozenge 0 = \theta, \forall \theta \in [0,1]$.

Definition 1.3. [5] The 5-tuple $(\Omega, \mathcal{M}, \mathcal{N}, *, \lozenge)$ is called an IFMSs if Ω is any non-empty set, $*$ is a CTN, $\lozenge$ is a CTCN, and $\mathcal{M}, \mathcal{N}$ are FSs on $\Omega^2 \times (0, +\infty)$ fulfilling the following circumstances: for all $\varpi, \omega, \lambda \in \Omega$, and $s, \tau > 0$

(i) $\mathcal{M}(\varpi,\omega,\tau) + \mathcal{N}(\varpi,\omega,\tau) \leq 1$;
(ii) $\mathcal{M}(\varpi,\omega,\tau)) > 0$;
(iii) $\mathcal{M}(\varpi,\omega,\tau) = 1 \Leftrightarrow \varpi = \omega$;
(iv) $\mathcal{M}(\varpi,\omega,\tau) = \mathcal{M}(\omega,\varpi,\tau)$;
(v) $\mathcal{M}(\varpi,\omega,\tau) * \mathcal{M}(\omega,\lambda,s) \leq \mathcal{M}(\varpi,\lambda,s+\tau)$;

(vi) $\mathcal{M}(\varpi, \omega, .): (0, +\infty) \rightarrow [0,1]$ is continuous;
(vii) $\mathcal{N}(\varpi, \omega, \tau) > 0$;
(viii) $\mathcal{N}(\varpi, \omega, \tau) = 0 \Leftrightarrow \varpi = \omega$;
(ix) $\mathcal{N}(\varpi, \omega, \tau) = \mathcal{N}(\omega, \varpi, \tau)$;
(x) $\mathcal{N}(\varpi, \omega, \tau) \lozenge \mathcal{N}(\omega, \lambda, s) \geq \mathcal{N}(\varpi, \lambda, s + \tau)$;
(xi) $\mathcal{N}(\varpi, \omega, .): (0, +\infty) \rightarrow (0,1]$ is continuous.

It shows that $(\mathcal{M}, \mathcal{N})$ is an IFM on Ω.

Definition 1.4. [20] The 6-tuple $(\Omega, \mathcal{M}, \mathcal{N}, \mathcal{O}, *, \lozenge)$ is called an NMS if Ω is any non-empty set, $*$ is a CTN, $\lozenge$ is a CTCN, and $\mathcal{M}, \mathcal{N}, \mathcal{O}$ are neutrosophic sets on $\Omega^2 \times (0, +\infty)$ fulfilling the following axioms: for all $\varpi, \omega, \lambda \in \Omega$ and $s, \tau > 0$

(i) $\mathcal{M}(\varpi, \omega, \tau) + \mathcal{N}(\varpi, \omega, \tau) + \mathcal{O}(\varpi, \omega, \tau) \leq 3$;
(ii) $\mathcal{M}(\varpi, \omega, \tau)) > 0$;
(iii) $\mathcal{M}(\varpi, \omega, \tau) = 1 \Leftrightarrow \varpi = \omega$;
(iv) $\mathcal{M}(\varpi, \omega, \tau) = \mathcal{M}(\omega, \varpi, \tau)$;
(v) $\mathcal{M}(\varpi, \omega, \tau) * \mathcal{M}(\omega, \lambda, s) \leq \mathcal{M}(\varpi, \lambda, s + \tau)$;
(vi) $\mathcal{M}(\varpi, \omega, .): (0, +\infty) \rightarrow [0,1]$ is continuous;
(vii) $\mathcal{N}(\varpi, \omega, \tau) > 0$;
(viii) $\mathcal{N}(\varpi, \omega, \tau) = 0 \Leftrightarrow \varpi = \omega$;
(ix) $\mathcal{N}(\varpi, \omega, \tau) = \mathcal{N}(\omega, \varpi, \tau)$;
(x) $\mathcal{N}(\varpi, \omega, \tau) \lozenge \mathcal{N}(\omega, \lambda, s) \geq \mathcal{N}(\varpi, \lambda, s + \tau)$;
(xi) $\mathcal{N}(\varpi, \omega, .): (0, +\infty) \rightarrow (0,1]$ is continuous.
(xii) $\mathcal{O}(\varpi, \omega, \tau) > 0$;
(xiii) $\mathcal{O}(\varpi, \omega, \tau) = 0 \Leftrightarrow \varpi = \omega$;
(xiv) $\mathcal{O}(\varpi, \omega, \tau) = \mathcal{O}(\omega, \varpi, \tau)$;
(xv) $\mathcal{O}(\varpi, \omega, \tau) \lozenge \mathcal{O}(\omega, \lambda, s) \geq \mathcal{O}(\varpi, \lambda, s + \tau)$;
(xvi) $\mathcal{O}(\varpi, \omega, .): (0, +\infty) \rightarrow (0,1]$ is continuous.

Then, $(\mathcal{M}, \mathcal{N}, \mathcal{O})$ is a neutrosophic metric on Ω.

Definition 1.5. [21] A sequence $\{\varpi_n\}$ in an NMS that converges to $\varpi \in \Omega$ if for each $\tau > 0$,

$$\lim_{n \to +\infty} \mathcal{M}(\varpi_n, \varpi, \tau) = 1,$$

$$\lim_{n \to +\infty} \mathcal{N}(\varpi\_n, \varpi, \tau) = 0,$$

$$\lim_{n \to +\infty} \mathcal{O}(\varpi_n, \varpi, \tau) = 0.$$

Then, $\{\varpi_n\}$ is said to be convergent.

Definition 1.6. [21] A sequence $\{\varpi_n\}$ in an NMS. It is called Cauchy if and only if for each $\varkappa \in (0,1)$ and $\tau > 0$ there exists $n_0 \in \mathbb{N}$, such that

$\mathcal{M}(\varpi_n, \varpi_m, \tau) > 1 - \varkappa,$

$\mathcal{N}(\varpi_n, \varpi_m, \tau) < \varkappa,$

$\mathcal{O}(\varpi_n, \varpi_m, \tau) < \varkappa.$

Then, for all $n, m \geq n_0$.

Definition 1.7. [21] An NMS $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ is complete if and only if every Cauchy sequence in Ω is convergent.

Definition 1.8. [22] Let ζ and g be two self-mappings which are weakly compatible if they commute at their coincidence points.

Definition 1.9. [7] A mapping $\psi : [0, +\infty) \to [0, +\infty)$ is an ADF or CF fulfilling the following circumstances:

(i) ψ is monotonically increasing and continuous;
(ii) $\psi(\tau) = 0 \Leftrightarrow \tau = 0.$

Furthermore, Khan et al. [7] utilized the concept of CF to demonstrate the following conclusion.

Theorem 1.1. [5] Suppose (Ω, d) be a complete metric space, $\psi : [0, +\infty) \to [0, +\infty)$ be an ADF, and suppose $\zeta: \Omega \to \Omega$ be a self-mapping which satisfies the following inequality:

$$\psi\left(d(\zeta\varpi, \zeta\omega)\right) \leq c\, \psi\left(d(\varpi, \omega)\right)$$

for all $\varpi, \omega \in \Omega$ and for some $0 < c < 1$. Then, ζ has a unique FP.

Definition 1.10. [24] Let $(\Omega, \mathcal{M}, *)$ be an FMS. The mapping $\zeta: \Omega \to \Omega \times \Omega$ is said to be fuzzy contractive if there exists $\Upsilon \in (0,1)$, such that

$$\frac{1}{\mathcal{M}(\zeta(\varpi), \zeta(\omega), \tau)} - 1 \leq \Upsilon\left(\frac{1}{\mathcal{M}(\varpi, \omega, \tau)} - 1\right)$$

for each $\varpi, \omega \in \Omega$, and $\tau > 0$.

Definition 1.11. A function $\Phi : \mathbb{R} \to \mathbb{R}^+$ verifies the condition $*$ if the following axioms hold:

(i) $\Phi(\tau) = 0$ if and only if $\tau = 0$;

(ii) $\Phi\ (\tau)$ is increasing and $\Phi(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$;
(iii) Φ is left continuous in $(0, +\infty)$;
(iv) Φ is continuous at 0.

2. MAIN THEOREMS

In this section, some important results are proved in the sense of NMS. Also, some non-trivial examples are discussed.

Definition 2.1. A sequence $\{\varpi_n\}$ in an NMS $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ is called G-Cauchy, if

$$\lim_{n\to+\infty} \mathcal{M}(\varpi_n, \varpi_{n+m}, \tau) = 1,$$
$$\lim_{n\to+\infty} \mathcal{N}(\varpi_n, \varpi_{n+m}, \tau) = 0,$$
$$\lim_{n\to+\infty} \mathcal{O}(\varpi_n, \varpi_{n+m}, \tau) = 0,$$

and

$\forall\, m \in \mathbb{N}, \tau > 0.$

Definition 2.2. An NMS $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ is called G-Complete if every G-Cauchy sequence in Ω is convergent.

Definition 2.3. A pair of self-mappings (ζ, g) of an NMS $(\Omega, \mathcal{M}, \mathcal{N}, \mathcal{O}, *, \Diamond)$ is called compatible if

$$\lim_{n\to+\infty} \mathcal{M}(\zeta g\varpi_n, g\zeta\varpi_n, \tau) = 1,$$

$$\lim_{n\to+\infty} \mathcal{N}(\zeta g\varpi_n, g\zeta\varpi_n, \tau) = 0,$$

$$\lim_{n\to+\infty} \mathcal{O}(\zeta g\varpi_n, g\zeta\varpi_n, \tau) = 0.$$

For every $\tau > 0$, whenever $\{\varpi_n\}$ is a sequence in Ω such that $\lim_{n\to+\infty} \zeta\,\varpi_n = \lim_{n\to+\infty} g\,\varpi_n = \lambda$ for some $\lambda \in \Omega$.

Definition 2.4. Let $(\Omega, \mathcal{M}, \mathcal{N}, \mathcal{O}, *, \Diamond)$ be an NMS. The mapping $\zeta: \Omega \rightarrow \Omega \times \Omega$ is called neutrosophic contractive if $\exists\, \Upsilon \in (0,1)$, such that

$$\frac{1}{\mathcal{M}(\zeta(\varpi), \zeta(\omega), \tau)} - 1 \leq \Upsilon\left(\frac{1}{\mathcal{M}(\varpi, \omega, \tau)} - 1\right),$$

$$\mathcal{N}(\zeta(\varpi), \zeta(\omega), \tau) \leq \Upsilon \mathcal{N}(\varpi, \omega, \tau),$$

$$\mathcal{O}(\zeta(\varpi), \zeta(\omega), \tau) \leq \Upsilon \mathcal{O}(\varpi, \omega, \tau),$$

for each $\varpi, \omega\ \Omega\ \in \Omega$, and $\tau > 0$.

Definition 2.5. Suppose $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ be an NMS. The mappings $\zeta, \xi: \Omega \to \Omega$ are called neutrosophic (Φ, ψ)-weak contractions with respect to ζ if $\exists\ \psi: [0, +\infty) \to [0, +\infty)$, with $\psi(\varpi) > 0$ for $\varpi > 0$, and $\psi(0) = 0$ and the ADF Φ, such that

$$\Phi\left(\frac{1}{\mathcal{M}(\xi\varpi, \xi\omega, \tau)} - 1\right) \leq \Phi\left(\frac{1}{\mathcal{M}(\zeta\varpi, \zeta\omega, \tau)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}(\zeta\varpi, \zeta\omega, \tau)} - 1\right),$$

$$\Phi\big(\mathcal{N}(\xi\varpi, \xi\omega, \tau)\big) \leq \Phi\big(\mathcal{N}(\zeta\varpi, \zeta\omega, \tau)\big) - \psi\big(\mathcal{N}(\zeta\varpi, \zeta\omega, \tau)\big),$$

$$\Phi\big(\mathcal{O}(\xi\varpi, \xi\omega, \tau)\big) \leq \Phi\big(\mathcal{O}(\zeta\varpi, \zeta\omega, \tau)\big) - \psi\big(\mathcal{O}(\zeta\varpi, \zeta\omega, \tau)\big).$$

It holds for every $\varpi, \omega \in \Omega$, and each $\tau > 0$. Suppose ζ is identity mapping, then the mapping ξ is called neutrosophic (Φ, ψ)-weak contraction.

Theorem 2.1. Suppose $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ be a G-complete NMS and the mappings are $\zeta, \xi: \Omega \to \Omega$, such that

i) the range of $\xi\Omega$ is contained in the range of $\zeta\Omega$;
ii) for $\tau > 0, 0 < c < 1$, the contraction

$$\left(\frac{1}{\mathcal{M}\big(\xi\varpi, \xi\omega, \Phi(c\tau)\big)} - 1\right) \leq \sigma\left(\frac{1}{\mathcal{M}\big(\zeta\varpi, \zeta\omega, \Phi(\tau)\big)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\big(\zeta\varpi, \zeta\omega\Phi(\tau)\big)} - 1\right), \tag{2.1}$$

$$\big(\mathcal{N}(\xi\varpi, \xi\omega, \Phi(c\tau)\big) \leq \sigma\Big(\mathcal{N}\big(\zeta\varpi, \zeta\omega, \Phi(\tau)\big)\Big) - \psi\Big(\mathcal{N}\big(\zeta\varpi, \zeta\omega, \Phi(\tau)\big)\Big), \tag{2.2}$$

$$\big(\mathcal{O}(\xi\varpi, \xi\omega, \Phi(c\tau)\big) \leq \sigma\Big(\mathcal{O}\big(\zeta\varpi, \zeta\omega, \Phi(\tau)\big)\Big) - \psi\Big(\mathcal{O}\big(\zeta\varpi, \zeta\omega, \Phi(\tau)\big)\Big), \tag{2.3}$$

It holds for all $\varpi, \omega \in \Omega$, where $\mathcal{M}(\zeta\varpi, \zeta\omega, \Phi(\tau)) > 0$ and $\mathcal{N}(\zeta\varpi, \zeta\omega, \Phi(\tau)) > 0$ φ and Φ fulfill the Definition 1.11. Moreover, for the ADF ψ and α, $\tau - \sigma(\tau) + \psi(\tau) > 0$ and $(\sigma - \psi)^n(\theta_n) \to 0$, $\theta_n \to 0$ as $n \to +\infty$; if $\zeta\Omega$ is G-complete subspace of Ω, then there exists a coincidence point of ζ and ξ.

Proof: Take an arbitrary element $\varpi_0 \in \Omega$ and utilizing the condition (i), let $\{\omega_n\}$ be a sequence, such that $\omega_n = \xi\varpi_n = \zeta\varpi_{n+1}$, assuming that $\{\omega_n\}$ is Cauchy. Further, if $\omega_{n-1} = \omega_n$, then there is a coincidence point of ξ and ζ. Suppose $\omega_{n-1} \neq \omega_n$ for all $\geq$, which implies that

$\mathcal{M}\big(\ \omega_{n-1},\ \omega_{n,}\tau\big) \neq 1,$

$\mathcal{N}\big(\ \omega_{n-1},\ \omega_{n,}\tau\big) \neq 0,$

$\mathcal{O}\big(\ \omega_{n-1},\ \omega_{n,}\tau\big) \neq 0.$

That is,

$$\mathcal{M}\big(\xi\varpi_{n-1},\ \xi\varpi_{n,}\tau\big) \neq 1\ \forall\ n \geq 1, \forall\ \tau > 0, \tag{2.4}$$

$$\mathcal{N}\big(\xi\varpi_{n-1},\ \xi\varpi_{n,}\tau\big) \neq 0\ \forall\ n \geq 1, \forall\ \tau > 0, \tag{2.5}$$

$$\mathcal{O}\big(\xi\varpi_{n-1},\ \xi\varpi_{n,}\tau\big) \neq 0\ \forall\ n \geq 1, \forall\ \tau > 0. \tag{2.6}$$

assuming that if possible for some n,

$$\left(\frac{1}{\mathcal{M}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)\big)} - 1\right) \leq \left(\frac{1}{\mathcal{M}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\big)} - 1\right),$$
$$\mathcal{N}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)\big) \leq \mathcal{N}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\big),$$
$$\mathcal{O}(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)) \leq \mathcal{O}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\big).$$

By substituting $\varpi = \varpi_{n,}$, $\omega = \varpi_{n+1}$ in inequalities (**2.1**), (**2.2**), and (**2.3**), we get

$$\left(\frac{1}{\mathcal{M}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\big)} - 1\right)$$
$$\leq \sigma\left(\frac{1}{\mathcal{M}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big)} - 1\right),$$
$$= \sigma\left(\frac{1}{\mathcal{M}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)\big)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)\big)} - 1\right), \tag{2.7}$$

$$\mathcal{N}(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)) \leq \sigma(\mathrm{N}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big) - \psi(\mathrm{N}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big)$$
$$= \sigma(\mathcal{N}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(\tau)\big) - \psi(\mathrm{N}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(\tau)\big), \tag{2.8}$$

$$\mathcal{O}(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)) \leq \sigma(\mathcal{O}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big) - \psi(\mathcal{O}\big(\zeta\varpi_n, \zeta\varpi_{n+1}, \Phi(c\tau)\big)$$
$$= \sigma(\mathcal{O}\big(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(\tau)\big) - \psi(\mathcal{O}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(\tau)\big). \tag{2.9}$$

However, $\tau - \alpha(\tau) + \psi(\tau) > 0$ together with **Eq.** (**2.7**), (**2.8**), and (**2.9**) lead us to a contradiction. Thus, for all n

$$\frac{1}{\mathcal{M}(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau))} - 1 < \frac{1}{\mathcal{M}\big(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(c\tau)\big)} - 1, \tag{2.10}$$

$$\mathcal{N}\left(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\right) < \mathcal{N}\left(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(\tau)\right), \tag{2.11}$$

$$\mathcal{O}\left(\xi\varpi_n, \xi\varpi_{n+1}, \Phi(c\tau)\right) < \mathcal{O}\left(\xi\varpi_{n-1}, \xi\varpi_n, \Phi(\tau)\right). \tag{2.12}$$

With the circumstances of Φ, one may have $\tau > 0$, then

$$\mathcal{M}(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)) > 0,$$

$$\mathcal{N}\left(\zeta\varpi, \zeta\omega, \Phi(\tau)\right) > 0,$$

$$\mathcal{O}\left(\zeta\varpi, \zeta\omega, \Phi(\tau)\right) > 0.$$

Therefore, using inequalities (**2.1**), (**2.2**), and (**2.3**), we get

$$\frac{1}{\mathcal{M}\left(\omega_0, \omega_1, \Phi(c\tau)\right)} - 1 = \left(\frac{1}{\mathcal{M}\left(\xi\varpi_0, \xi\varpi_1, \Phi(c\tau)\right)} - 1\right) \leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(c\tau)\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(c\tau)\right)} - 1\right),$$

$$\mathcal{N}\left(\omega_0, \omega_1, \Phi(c\tau)\right) = \mathcal{N}\left(\xi\varpi_0, \xi\varpi_1, \Phi(c\tau)\right) \leq \alpha(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right) - \psi(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right),$$

$$\mathcal{O}\left(\omega_0, \omega_1, \Phi(c\tau)\right) = \mathcal{O}\left(\xi\varpi_0, \xi\varpi_1, \Phi(c\tau)\right) \leq \alpha(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right) - \psi(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right).$$

In view of inequalities (**2.10**), (**2.11**), and (**2.12**) the above expression become

$$\frac{1}{\mathcal{M}\left(\xi\varpi_1, \xi\varpi_2, \Phi(c\tau)\right)} - 1 \leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right)} - 1\right) \tag{2.13}$$

$$\mathcal{N}\left(\xi\varpi_1, \xi\varpi_2, \Phi(c\tau)\right) \leq \alpha(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right) - \psi(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right), \tag{2.14}$$

$$\mathcal{O}\left(\xi\varpi_1, \xi\varpi_2, \Phi(c\tau)\right) \leq \alpha(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right) - \psi(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right). \tag{2.15}$$

Again

$$\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\right) > 0,$$

which implies

$$\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{1}{c}\right)\right) > 0$$

$$\mathcal{N}\big(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\big) > 0,$$

which implies

$$\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{1}{c}\right)\right) > 0,$$

$$\mathcal{O}\big(\zeta\varpi_1, \zeta\varpi_2, \Phi(\tau)\big) > 0,$$

which implies

$$\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{1}{c}\right)\right) > 0.$$

Therefore, by the application of inequality (**2.1**), we obtain

$$\frac{1}{\mathcal{M}\big(\omega_0, \omega_1, \Phi(\tau)\big)} - 1 = \left(\frac{1}{\mathcal{M}\big(\xi\varpi_0, \xi\varpi_1, \Phi(c\tau)\big)} - 1\right)$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right)} - 1\right)$$

$$\mathcal{N}\big(\omega_0, \omega_1, \Phi(\tau)\big) = \mathcal{N}\big(\xi\varpi_0, \xi\varpi_1, \Phi(\tau)\big)$$
$$\leq \alpha(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right) - \psi(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right),$$

$$\mathcal{O}\big(\omega_0, \omega_1, \Phi(\tau)\big) = \mathcal{O}\big(\xi\varpi_0, \xi\varpi_1, \Phi(\tau)\big)$$
$$\leq \alpha(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right) - \psi(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right).$$

Again, using inequalities (**2.10**), (**2.11**), and (**2.12**) the above equations turn out to be

$$\frac{1}{\mathcal{M}\big(\xi\varpi_1, \xi\varpi_2, \Phi(\tau)\big)} - 1$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right)} - 1\right)$$
$$- \psi\left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right)} - 1\right) \qquad (2.16)$$

$$\mathcal{N}\left(\xi\varpi_1, \xi\varpi_2, \Phi(\tau)\right) \leq \alpha(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right) - \psi(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right), \tag{2.17}$$

$$\mathcal{O}\left(\xi\varpi_1, \xi\varpi_2, \Phi(\tau)\right) \leq \alpha(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right) - \psi(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c}\right)\right), \tag{2.18}$$

By continuing in this manner, one obtains n times

$$\frac{1}{\mathcal{M}\left(\omega_{n-1}, \omega_n, \Phi(\tau)\right)} - 1 \leq (\sigma - \psi)^n \left(\frac{1}{\mathcal{M}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c^n}\right)\right)} - 1 \right), \tag{2.19}$$

$$\mathcal{N}\left(\omega_{n-1}, \omega_n, \Phi(\tau)\right) \leq (\sigma - \psi)^n \left(\mathcal{N}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c^n}\right)\right)\right), \tag{2.20}$$

$$\mathcal{O}\left(\omega_{n-1}, \omega_n, \Phi(\tau)\right) \leq (\sigma - \psi)^n \left(\mathcal{O}\left(\zeta\varpi_1, \zeta\varpi_2, \Phi\left(\frac{\tau}{c^n}\right)\right)\right). \tag{2.21}$$

Then, using the assumption (ii) which implies that

$\mathcal{M}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi(c\tau)\right) > 0,$

$\mathcal{N}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi(c\tau)\right) > 0,$

$\mathcal{O}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi(c\tau)\right) > 0.$

Similarly, one might get

$$\frac{1}{\mathcal{M}\left(\omega_{n-1}, \omega_n, \Phi(\tau)\right)} - 1 \leq (\sigma - \psi)^n \left(\frac{1}{\mathcal{M}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi\left(\frac{c\tau}{c^n}\right)\right)} - 1 \right), \tag{2.22}$$

$$\mathcal{N}\left(\omega_{n-1}, \omega_n, \Phi(c\tau)\right) \leq (\sigma - \psi)^n \, \mathcal{N}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi\left(\frac{c\tau}{c^n}\right)\right), \tag{2.23}$$

$$\mathcal{O}\left(\omega_{n-1}, \omega_n, \Phi(c\tau)\right) \leq (\sigma - \psi)^n \, \mathcal{O}\left(\zeta\varpi_2, \zeta\varpi_3, \Phi\left(\frac{c\tau}{c^n}\right)\right). \tag{2.24}$$

Furthermore, for $n > \varkappa$, we have

$$\frac{1}{\mathcal{M}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right)} - 1 \leq (\sigma - \psi)^{n-\varkappa+1} \left(\frac{1}{\mathcal{M}\left(\zeta\varpi_{\varkappa+1}, \zeta\varpi_{\varkappa+2}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right)} - 1 \right), \tag{2.25}$$

$$\mathcal{N}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \leq (\sigma - \psi)^{n-\varkappa+1}\, \mathcal{N}\left(\zeta\varpi_{\varkappa+1}, \zeta\varpi_{\varkappa+2}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right), \tag{2.26}$$

$$\mathcal{O}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \leq (\sigma - \psi)^{n-\varkappa+1}\, \mathcal{O}\left(\zeta\varpi_{\varkappa+1}, \zeta\varpi_{\varkappa+2}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right). \tag{2.27}$$

Assuming $\omega_n = \zeta\varpi_{n+1}$ and using the inequalities (**2.25**), (**2.26**), and (**2.27**), we get

$$\frac{1}{\mathcal{M}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right)} - 1 \leq (\sigma - \psi)^{n-\varkappa+1} \left(\frac{1}{\mathcal{M}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right)} - 1 \right),$$

$$(\sigma - \psi)^{n-\varkappa+1}\,(\theta_n), \text{where } \theta_n = \left(\frac{1}{\mathcal{M}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right)} - 1 \right), \tag{2.28}$$

$$\mathcal{N}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \leq (\sigma - \psi)^{n-\varkappa+1} \mathcal{N}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right),$$

$$= (\sigma - \psi)^{n-\varkappa+1}\,(\partial_n), \text{where } \partial_n = \mathcal{N}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right), \tag{2.29}$$

$$\mathcal{O}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \leq (\sigma - \psi)^{n-\varkappa+1} \mathcal{O}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right),$$

$$= (\sigma - \psi)^{n-\varkappa+1}\,(\partial_n), \text{where } \partial_n = \mathcal{O}\left(\omega_{\varkappa}, \omega_{\varkappa+1}, \Phi\left(\frac{c^{\varkappa}\tau}{c^{n-\varkappa+1}}\right)\right). \tag{2.30}$$

Considering the assumption, $(\sigma - \psi)^n\,(\theta_n) \to 0$, therefore, $(\theta_n) \to 0$ as $n \to +\infty$. $\forall\, \varkappa > 0$

$$\mathcal{M}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \to 1 \text{ as } n \to +\infty. \tag{2.31}$$

Likewise, $(\sigma - \psi)^n\,(\partial_n) \to 0$, whenever $(\partial_n) \to 0$ as $n \to +\infty\ \forall\, \varkappa > 0$

$$\mathcal{N}\left(\omega_{n-1}, \omega_n, \Phi(c^{\varkappa}\tau)\right) \to 0 \text{ as } n \to +\infty. \tag{2.32}$$

Also, $(\sigma - \psi)^n(\partial_n) \to 0$, whenever $(\partial_n) \to 0$ as $n \to +\infty$. Furthermore, $\forall\, \varkappa > 0$

$$\mathcal{O}(\,\omega_{n-1},\omega_n,\Phi(c^{\varkappa}\tau)) \to 0 \text{ as } n \to +\infty. \tag{2.33}$$

For $\tau > 0$, one may discover $\varkappa > 0$, such that $\Phi(c^{\varkappa}\tau) < \varepsilon$. From (**2.31**), we obtain that $\mathcal{M}\,(\,\omega_{n-1},\omega_n,\varepsilon) \to 1$ as $n \to +\infty$ or

$$\mathcal{M}\,(\,\omega_n,\omega_{n+1},\varepsilon) \to 1 \text{ as } n \to +\infty \tag{2.34}$$

and $\mathcal{N}\,(\,\omega_{n-1},\omega_n,\varepsilon) \to 1$ as $n \to +\infty$ or

$$\mathcal{N}\,(\,\omega_n,\omega_{n+1},\varepsilon) \to 1 \text{ as } n \to +\infty \tag{2.35}$$

Similarly, as $\mathcal{O}(\,\omega_{n-1},\omega_n,\varepsilon) \to 1$ as $n \to +\infty$ or

$$\mathcal{O}(\,\omega_n,\omega_{n+1},\varepsilon) \to 1 \text{ as } n \to +\infty \tag{2.36}$$

Using the triangular inequality, we have

$$\mathcal{M}\,(\,\omega_n,\omega_{n+\varsigma},\varepsilon) \geq \mathcal{M}\,\left(\,\omega_n,\omega_{n+1},\frac{\varepsilon}{\varsigma}\right) * \mathcal{M}\,\left(\,\omega_{n+1},\omega_{n+2},\frac{\varepsilon}{\varsigma}\right) * \ldots * \mathcal{M}\,\left(\,\omega_{n+\varsigma},\omega_{n+\varsigma+1},\frac{\varepsilon}{\varsigma}\right),$$

$$\mathcal{N}(\,\omega_n,\omega_{n+\varsigma},\varepsilon) \leq \mathcal{N}\left(\,\omega_n,\omega_{n+1},\frac{\varepsilon}{\varsigma}\right) \lozenge\, \mathcal{N}\left(\,\omega_{n+1},\omega_{n+2},\frac{\varepsilon}{\varsigma}\right) \lozenge \ldots \lozenge\, \mathcal{N}\left(\,\omega_{n+\varsigma},\omega_{n+\varsigma+1},\frac{\varepsilon}{\varsigma}\right),$$

$$\mathcal{O}(\,\omega_n,\omega_{n+\varsigma},\varepsilon) \leq \mathcal{O}\left(\,\omega_n,\omega_{n+1},\frac{\varepsilon}{\varsigma}\right) \lozenge\, \mathcal{O}\left(\,\omega_{n+1},\omega_{n+2},\frac{\varepsilon}{\varsigma}\right) \lozenge \ldots \lozenge\, \mathcal{O}\left(\,\omega_{n+\varsigma},\omega_{n+\varsigma+1},\frac{\varepsilon}{\varsigma}\right)$$

As limit $n \to +\infty$ in the above inequalities and also using (**2.34**), (**2.35**), and (**2.36**), we get

$$\mathcal{M}\,\left(\,\omega_n,\omega_{n+\varsigma},\varepsilon\right) \to 1,$$

$$\mathcal{N}\left(\,\omega_n,\omega_{n+\varsigma},\varepsilon\right) \to 0,$$

$$\mathcal{O}\left(\,\omega_n,\omega_{n+\varsigma},\varepsilon\right) \to 0.$$

Its implies that $\{\omega_n\}$ is a G-Cauchy sequence. Since $(\Omega,\mathcal{M},\mathcal{N},\mathcal{O},*,\lozenge)$ is G-complete, then $\{\omega_n\}$ is convergent, and hence $\exists\,\lambda \in \Omega$, such that $\omega_n \to \mathrm{z}\,\theta s\; n \to +\infty$,

$$\omega_n = \xi\varpi_n = \zeta\varpi_{n+1} \to \lambda.$$

Suppose that $v \in \Omega$, such that $\zeta v = \lambda$. Here, we find that v is the coincidence point of ξ and ζ. However, its sufficient that $\xi v = \lambda$. Then,

$$\mathcal{M}\,(\,\xi v,\lambda,\varepsilon) \geq \mathcal{M}\,\left(\,\xi v,\omega_n,\frac{\varepsilon}{2}\right) * \mathcal{M}\,\left(\,\omega_n,\lambda,\frac{\varepsilon}{2}\right). \tag{2.37}$$

Using the Definition 1.11, we can find a $\tau_2 > 0$, such that $\varphi(\tau_2) < \frac{\varepsilon}{2}$. Since $\omega_n \to z$ as $n \to +\infty$, then there exists $\varepsilon > 0$ for all $n > m$, such that $\mathcal{M}\left(\omega_n, \lambda, \varphi(\tau_2)\right) > 0$. Therefore, by contraction (2) we have $n > m$

$$\frac{1}{\mathcal{M}\left(\xi v, \omega_n, \frac{\varepsilon}{2}\right)} - 1 = \frac{1}{\mathcal{M}\left(\xi v, \xi\varpi_n, \varphi(\tau_2)\right)} - 1$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta v, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_2}{c}\right)\right)} - 1\right)$$
$$- \psi\left(\frac{1}{\mathcal{M}\left(\zeta v, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_2}{c}\right)\right)} - 1\right).$$

On using inequality (**2.10**), we obtain

$$\frac{1}{\mathcal{M}\left(\xi v, \xi\varpi_{n+1}, \varphi(\tau_2)\right)} - 1$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta v, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_2}{c}\right)\right)} - 1\right)$$
$$- \psi\left(\frac{1}{\mathcal{M}\left(\zeta v, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_2}{c}\right)\right)} - 1\right).$$

As limit $n \to +\infty, \Phi(0) = 0$ use the continuity of ψ and α, one has

$$\mathcal{M}\left(\xi v, \omega_n, \frac{\varepsilon}{2}\right) \to 1 \text{ as } n \to +\infty. \tag{2.38}$$

Taking $n \to +\infty$ inequality (**2.37**) and using (**2.38**) with the continuity of functions ψ, α, and the fact that $\omega_n \to z$ as $n \to +\infty$, we get $\mathcal{M}(\xi v, \omega_n, \varepsilon) = 1$. In a similar way, we can find $\mathcal{N}(\xi v, \omega_n, \varepsilon) = 0$ and $\mathcal{O}(\xi v, \omega_n, \varepsilon) = 0$ for every $\varepsilon > 0$. It follows that

$$\xi v = \lambda. \tag{2.39}$$

Thus,

$$\zeta v = \xi v = \lambda.$$

Theorem 2.2. Let the assumptions used in the Theorem 2.1, then the coincidence point λ becomes the CFP of ξ and ζ which is unique if the pair (ζ, ξ) is weakly compatible.

Proof: Using **Eq. (40)**, we have the coincidence point v and $\zeta v = \xi v = \lambda$. So, the pair (ζ, ξ) is weakly compatible, thus

$\zeta\lambda = \zeta\xi v = \xi\zeta v = \xi\lambda.$

It is to be demonstrated this λ is a CFP of ξ and ζ. Again,

$$\mathcal{M}\ (\xi\lambda, \lambda, \varepsilon) \geq \mathcal{M}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) * \mathcal{M}\left(\omega_n, \lambda, \frac{\varepsilon}{2}\right), \tag{2.40}$$

$$\mathcal{N}\ (\xi\lambda, \lambda, \varepsilon) \leq \mathcal{N}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) \Diamond\, \mathcal{N}\left(\omega_n, \lambda, \frac{\varepsilon}{2}\right), \tag{2.41}$$

$$\mathcal{O}(\xi\lambda, \lambda, \varepsilon) \leq \mathcal{O}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) \Diamond\, \mathcal{O}\left(\omega_n, \lambda, \frac{\varepsilon}{2}\right). \tag{2.42}$$

From the property of Φ -function, one may find a $\tau_3 > 0$, such as $(\tau_3) < \frac{\varepsilon}{2}$ and $\omega_n \to \mathrm{z}$ as $n \to +\infty$, hence $\exists\, m \in \mathcal{N}\ \forall\, n > m$,

$\mathcal{M}(\omega_n, \lambda, (\tau_3)) > 0,$

$\mathcal{N}\big(\omega_n, \lambda, (\tau_3)\big) > 0,$

$\mathcal{O}\big(\omega_n, \lambda, (\tau_3)\big) > 0.$

Then, for $n > m$,

$$\frac{1}{\mathcal{M}\ \left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right)} - 1 = \frac{1}{\mathcal{M}\ (\xi\lambda, \xi\varpi_n, \varphi(\tau_3))} - 1$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\ \left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right)} - 1\right)$$
$$- \psi\left(\frac{1}{\mathcal{M}\ \left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right)} - 1\right).$$

This is obvious from inequality (**2.1**). Applying inequalities (**2.10**), (**2.11**), and (**2.12**), we get

$$\frac{1}{\mathcal{M}\left(\xi\lambda, \xi\varpi_n, \varphi(\tau_3)\right)} - 1 \leq \alpha\left(\frac{1}{\mathcal{M}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right)} - 1\right),$$

and

$$\mathcal{N}\left(\xi\lambda, \xi\varpi_n, \varphi(\tau_3)\right) \leq \alpha\mathcal{N}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right) - \psi\mathcal{N}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right),$$

$$\mathcal{O}\left(\xi\lambda, \xi\varpi_n, \varphi(\tau_3)\right) \leq \alpha\,\mathcal{O}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right) - \psi\mathcal{O}\left(\zeta\lambda, \zeta\varpi_{n+1}, \Phi\left(\frac{\tau_3}{c}\right)\right)$$

Using the limit $n \to +\infty$ with the help of Definition 1.11, we have

$$\mathcal{M}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) \to 1 \text{ as } n \to +\infty, \tag{2.43}$$

$$\mathcal{N}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) \to 1 \text{ as } n \to +\infty, \tag{2.44}$$

and

$$\mathcal{O}\left(\xi\lambda, \omega_n, \frac{\varepsilon}{2}\right) \to 1 \text{ as } n \to +\infty. \tag{2.45}$$

Similarly, taking $n \to +\infty$ in inequalities (**2.40**), (**2.41**), and (**2.42**), using (**2.43**), (**2.44**), and (**2.45**) with the continuity of α and the fact that $\omega_n \to \mathrm{z}$ as $n \to +\infty$, it is clear that

$$\mathcal{M}(\xi\lambda, \lambda, \varepsilon) = 1, \forall \ \varepsilon > 0,$$

$$\mathcal{N}(\xi\lambda, \lambda, \varepsilon) = 0, \forall \ \varepsilon > 0,$$

$$\mathcal{O}(\xi\lambda, \lambda, \varepsilon) = 0, \forall \ \varepsilon > 0.$$

Thus, $\xi\lambda = \lambda$. We prove that $\zeta\lambda = \xi\lambda = \lambda$. Then, λ is a CFP of ζ and ξ . Finally, the uniqueness of the CPF is shown. Suppose λ' is any other FP. Then, it needs to be shown that $\lambda = \lambda'$

$$\mathcal{M}\left(\lambda, \lambda', \Phi(s)\right) > 0,$$

$\mathcal{N}\big(\lambda, \lambda', \Phi(s)\big) > 0,$

$\mathcal{O}\big(\lambda, \lambda', \Phi(s)\big) > 0.$

Using the inequality (**2.1**),

$$\frac{1}{\mathcal{M}(\lambda, \lambda', \Phi(cs))} - 1 = \frac{1}{\mathcal{M}\ \big(\xi\lambda, \xi\lambda', \Phi(cs)\big)} - 1$$
$$\leq \alpha\left(\frac{1}{\mathcal{M}\ \big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\ \big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)} - 1\right), \quad (2.46)$$

$$\mathcal{N}\big(\lambda, \lambda', \Phi(cs)\big) = \mathcal{N}\big(\xi\lambda, \xi\lambda', \Phi(cs)\big)$$
$$\leq \alpha\left(\mathcal{N}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right) - \psi\left(\mathcal{N}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right),$$

$$\leq \alpha\left(\mathcal{N}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right) - \psi\left(\mathcal{N}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right), \quad (2.47)$$

$$\mathcal{O}\big(\lambda, \lambda', \Phi(cs)\big) = \mathcal{O}\big(\xi\lambda, \xi\lambda', \Phi(cs)\big) \leq \alpha\left(\mathcal{O}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right) - \psi\left(\mathcal{O}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right),$$

$$\leq \alpha\left(\mathcal{O}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right) - \psi\left(\mathcal{O}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big)\right), \quad (2.48)$$

Also, $\mathcal{M}\ \big(\zeta\lambda, \zeta\lambda', \Phi(s)\big) > 0$ implies that $\mathcal{M}\ \left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{s}{c}\right)\right) > 0,$ $\mathcal{N}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big) > 0$ implies that $\mathcal{N}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{s}{c}\right)\right) > 0$, and $\mathcal{O}\big(\zeta\lambda, \zeta\lambda', \Phi(s)\big) > 0$ implies that $\mathcal{O}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{s}{c}\right)\right) > 0.$ Here, replacing s by $\frac{s}{c}$ in the above inequalities (**2.46**), (**2.47**), and (**2.48**), we get

$$\frac{1}{\mathcal{M}(\lambda, \lambda', \Phi(s))} - 1 \leq \alpha\left(\frac{1}{\mathcal{M}\ \left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{s}{c}\right)\right)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}\ \left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{s}{c}\right)\right)} - 1\right),$$

$$\mathcal{N}\big(\lambda, \lambda', \Phi(s)\big) \leq \alpha\left(\mathcal{N}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{S}{C}\right)\right)\right) - \psi\left(\mathcal{N}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{S}{C}\right)\right)\right),$$

and

$$\mathcal{O}\big(\lambda, \lambda', \Phi(s)\big) \leq \alpha\left(\mathcal{O}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{S}{C}\right)\right)\right) - \psi\left(\mathcal{O}\left(\zeta\lambda, \zeta\lambda', \Phi\left(\frac{S}{C}\right)\right)\right),$$

Repeating the above n-times, we get

$$\frac{1}{\mathcal{M}(\lambda,\lambda',\Phi(s))}-1 \leq (\sigma-\psi)^n\left(\frac{1}{\mathcal{M}\left(\zeta\lambda,\zeta\lambda',\Phi\left(\frac{s}{c}\right)\right)}-1\right),$$

$$=(\sigma-\psi)^n(\theta_n)$$

$$\mathcal{N}(\lambda,\lambda',\Phi(s)) \leq (\sigma-\psi)^n\left(\mathcal{N}\left(\zeta\lambda,\zeta\lambda',\Phi\left(\frac{s}{c}\right)\right)\right),$$

$$=(\sigma-\psi)^n(\partial_n)$$

$$\mathcal{O}(\lambda,\lambda',\Phi(s)) \leq (\sigma-\psi)^n\left(\mathcal{O}\left(\zeta\lambda,\zeta\lambda',\Phi\left(\frac{s}{c}\right)\right)\right),$$

$$=(\sigma-\psi)^n(\partial_n)$$

As $n \to +\infty$ $(\sigma-\psi)^n(\theta_n) \to 0$ and $(\sigma-\psi)^n(\partial_n) \to 0$, we have

$$\mathcal{M}(\lambda,\lambda',\Phi(s)) = 1, \forall\, s > 0, \mathcal{N}(\lambda,\lambda',\Phi(s)) = 0\,, \forall\, s > 0, \mathcal{O}(\lambda,\lambda',\Phi(s)) = 0\,, \forall\, s > 0.$$

Again, from inequalities (**2.46**), (**2.47**), and (**2.48**), it follows that $\mathcal{M}(\lambda,\lambda',\Phi(cs)) > 0, \mathcal{N}(\lambda,\lambda',\Phi(cs)) > 0,$ and $\mathcal{O}(\lambda,\lambda',\Phi(cs)) > 0.$ When the arguments are similar then s is changed by cs, we get

$$\mathcal{M}(\lambda,\lambda',\Phi(cs)) = 1, \mathcal{N}(\lambda,\lambda',\Phi(cs)) = 0, \mathcal{O}(\lambda,\lambda',\Phi(cs)) = 0.$$

Generally, $\mathcal{M}(\lambda,\lambda',\Phi(c^n s)) = 1, \mathcal{N}(\lambda,\lambda',\Phi(c^n s)) = 0$ and $\mathcal{O}(\lambda,\lambda',\Phi(c^n s)) = 0\ \forall\ n \in \mathcal{N} \cup \{0\}$. Clearly, for any given $\varepsilon > 0$ there exists $\varkappa \in \cup\, \mathbb{N}\, \{0\}$, such that $\Phi(c^{\varkappa} s) < \varepsilon$. From the foregoing analysis, we get $\mathcal{M}(\lambda,\lambda',\varepsilon) = 1$, $\mathcal{N}(\lambda,\lambda',\varepsilon) = 0$, and $\mathcal{O}(\lambda,\lambda',\varepsilon) = 0$. For all $\varepsilon > 0$, which implies that

$$\lambda = \lambda',$$

Hence, it completes the proof.

Corollary 2.1. Suppose $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ be a G-complete NMS and the mapping $\zeta, : \Omega \to \Omega$, such that for $\tau > 0\,, 0 < c < 1$, the contraction

$$\mathcal{M}(\zeta\varpi,\zeta\omega,\Phi(\tau)) \leq \mathcal{M}(\varpi,\omega,\tau),$$
$$\mathcal{N}(\zeta\varpi,\zeta\omega,\Phi(\tau)) \leq \mathcal{N}(\varpi,\omega,\tau),$$
$$\mathcal{O}(\zeta\varpi,\zeta\omega,\Phi(\tau)) \leq \mathcal{O}(\varpi,\omega,\tau),$$

holds for all $\varpi, \omega \in \Omega$, where $\mathcal{M}(\zeta\varpi, \zeta\omega, \Phi(\tau)) > 0, \mathcal{N}(\zeta\varpi, \zeta\omega, \Phi(\tau)) > 0$ and $\mathcal{O}(\zeta\varpi, \zeta\omega, \Phi(\tau)) > 0$, and Φ fulfill the **Definition 2.6.** If $\zeta\Omega$ is G -complete subspace of Ω, then there exists a coincidence point of ζ.

Example 2.1. Let $(\Omega, \mathcal{M}\ \mathcal{N}, \mathcal{O}, *, \Diamond)$ be a complete NMS, where $\Omega = \{\varpi_1, \varpi_2, \varpi_3\}$

$\theta \Diamond \partial = \max\{\theta, \partial\}$, $\theta * \partial = \min\{\theta, \partial\}$, and $\mathcal{M}(\varpi, \omega, \tau)$ be defined as

$$\mathcal{M}(\varpi_2, \varpi_3, \tau) = \mathcal{M}(\varpi_3, \varpi_2, \tau) = \begin{cases} 0, & \text{if } \tau = 0 \\ 0.8, & \text{if } 0 < \tau < 3 \\ 1, & \text{if } \tau \geq 3 \end{cases}$$

$$\mathcal{N}(\varpi_2, \varpi_3, \tau) = \mathcal{N}(\varpi_3, \varpi_2, \tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ \frac{1}{2}, & \text{if } 0 < \tau < 3 \\ 0, & \text{if } \tau \geq 3 \end{cases}$$

$$\mathcal{O}(\varpi_2, \varpi_3, \tau) = \mathcal{O}(\varpi_3, \varpi_2, \tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ \frac{1}{4}, & \text{if } 0 < \tau < 3 \\ 0, & \text{if } \tau \geq 3 \end{cases}$$

$$\mathcal{M}(\varpi_1, \varpi_3, \tau) = \mathcal{M}(\varpi_3, \varpi_1, \tau) = \mathcal{M}(\varpi_1, \varpi_2, \tau) = \mathcal{M}(\varpi_2, \varpi_1, \tau) = \begin{cases} 0, & \text{if } \tau = 0 \\ 1, & \text{if } \tau > 0 \end{cases}$$

$$\mathcal{N}(\varpi_1, \varpi_3, \tau) = \mathcal{N}(\varpi_3, \varpi_1, \tau) = \mathcal{N}(\varpi_1, \varpi_2, \tau) = \mathcal{N}(\varpi_2, \varpi_1, \tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ 0, & \text{if } \tau > 0 \end{cases}$$

$$\mathcal{O}(\varpi_1, \varpi_3, \tau) = \mathcal{O}(\varpi_3, \varpi_1, \tau) = \mathcal{O}(\varpi_1, \varpi_2, \tau) = \mathcal{O}(\varpi_2, \varpi_1, \tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ 0, & \text{if } \tau > 0. \end{cases}$$

The mappings $\xi, \zeta: \Omega \to \Omega$ are defined as $\xi(\varpi_1) = \varpi_1$; $\xi(\varpi_2) = \varpi_3$; $\xi(\varpi_3) = \varpi_1$; $\zeta(\varpi_1) = \varpi_1$; $\zeta(\varpi_2) = \varpi_3$; $\zeta(\varpi_3) = \varpi_2$; and if $\Phi(\tau) = \tau$, $\psi(\tau) = \frac{\tau}{6}$ and $\alpha(\tau) = \tau$ and $c = \frac{1}{2}$. Clearly, the circumstances of Theorem 2.1 are fulfilled and (ζ, ξ) are also weakly compatible. Hence, ϖ_1 is the unique CFP of ζ and ξ.

Example 2.2. Suppose $\Omega = [0,+\infty)$ and $\theta * \partial = \min\{\theta, \partial\}$, $\theta \lozenge \partial = \max\{\theta, \partial\}$, and

$$\mathcal{M}(\varpi,\omega,\tau) = \frac{\tau}{\tau + |\varpi - \omega|},$$

$$\mathcal{N}(\varpi,\omega,\tau) = \frac{|\varpi - \omega|}{\tau + |\varpi - \omega|},$$

$$\mathcal{O}(\varpi,\omega,\tau) = \frac{|\varpi - \omega|}{\tau}.$$

For all $\varpi, \omega \in \Omega$ and $\tau > 0$. Then, $(\Omega, \mathcal{M}\,\mathcal{N}, \mathcal{O}, *, \lozenge)$ is a complete NMS. Let $\xi, \zeta: \Omega \to \Omega$ be given respectively by the formulas $\xi\varpi = \frac{\varpi}{3}$ and $\zeta\varpi = \varpi$ for all $\varpi \in \Omega$. Let $\psi, \alpha: [0,+\infty) \to [0,+\infty)$ be given respectively by the following formulas:

if $\Phi(\tau) = \tau$, $\psi(\tau) = \frac{\tau}{6}$ and $\alpha(\tau) = \tau$ and $c = \frac{1}{2}$.

Then, clearly, $\xi\Omega \subseteq \zeta\Omega$. For all $\varpi, \omega \in [0,+\infty)$, and $\tau > 0$, inequality (**2.1**) reduces to

$$\frac{|\xi(\varpi) - \xi(\omega)|}{\frac{\tau}{2}} \leq \alpha\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau}\right) - \psi\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau}\right) \tag{2.49}$$

$$\frac{|\xi(\varpi) - \xi(\omega)|}{\frac{\tau}{2} + |\xi(\varpi) - \xi(\omega)|} \leq \alpha\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau + |\zeta(\varpi) - \zeta(\omega)|}\right) - \psi\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau + |\zeta(\varpi) - \zeta(\omega)|}\right) \tag{2.50}$$

$$\frac{|\xi(\varpi) - \xi(\omega)|}{\frac{\tau}{2}} \leq \alpha\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau}\right) - \psi\left(\frac{|\zeta(\varpi) - \zeta(\omega)|}{\tau}\right) \tag{2.51}$$

Since, $\xi(\varpi) - \xi(\omega) = \frac{1}{3}(\varpi - \omega)$ and $\zeta(\varpi) - \zeta(\omega) = (\varpi - \omega)$, so by substituting the values in inequalities (**2.49**), (**2.50**), and (**2.51**)

$$\frac{2}{3\tau}|(\varpi - \omega)| \leq \left(\frac{1}{\tau}|(\varpi - \omega)|\right) - \left(\frac{1}{6\tau}|(\varpi - \omega)|\right),$$

$$\frac{2}{3\tau}|(\varpi - \omega)| \leq \frac{5}{3\tau}|(\varpi - \omega)|.$$

$$\frac{|\varpi - \omega|}{\frac{2}{3\tau} + |(\varpi - \omega)|} \leq \frac{|\varpi - \omega|}{\tau + |\varpi - \omega|} - \frac{|\varpi - \omega|}{\frac{\tau}{6} + |\varpi - \omega|}.$$

and

$$\frac{|\varpi-\omega|}{\frac{2}{3\tau}} \leq \frac{|\varpi-\omega|}{\tau} - \frac{|\varpi-\omega|}{\frac{\tau}{6}},$$

$$\frac{3\tau}{2}|\varpi-\omega| \leq \frac{|\varpi-\omega|}{\tau} - \frac{6}{\tau}|\varpi-\omega|.$$

Hence, all circumstances of Theorem 2.1 are fulfilled and 0 is a coincident point. Moreover, (ξ,ζ) is a weakly compatible pair of mappings, thus the coincidence point is also the CFP of ξ and ζ.

Example 2.3. Suppose $\Omega = [0,10]$ and $\theta * \partial = \min\{\theta, \partial\}$, $\theta \lozenge \partial = \max\{\theta, \partial\}$ and

$$\mathcal{M}(\varpi,\omega,\tau) = \frac{\tau}{\tau + \max\{\varpi,\omega\}},$$

$$\mathcal{N}(\varpi,\omega,\tau) = \frac{\max\{\varpi,\omega\}}{\tau + \max\{\varpi,\omega\}},$$

$$\mathcal{O}(\varpi,\omega,\tau) = \frac{\max\{\varpi,\omega\}}{\tau}.$$

For all $\varpi,\omega \in \Omega$ and $\tau > 0$. Then, $(\Omega, \mathcal{M}\,\mathcal{N}, \mathcal{O}, *, \lozenge)$ is a complete NMS. Let $\xi,\zeta: \Omega \to \Omega$ be given respectively by the formulas $\xi\varpi = \frac{\varpi}{3}$ and $\zeta\varpi = \varpi$ for all $\varpi \in \Omega$. Let $\psi,\alpha: [0,+\infty) \to [0,+\infty)$ be given respectively by the formulas. If $\Phi(\tau) = \tau$, $\psi(\tau) = \frac{\tau}{6}$ and $\alpha(\tau) = \tau$ and $c = \frac{1}{2}$, then $\xi\Omega \subseteq \zeta\Omega$. All circumstances of Theorem 2.2 are fulfilled and 0 is a coincident point. Moreover, (ξ,ζ) is a weakly compatible pair of mappings, thus the coincidence point is also the CFP of ξ and ζ.

3. APPLICATION TO FREDHOLM INTEGRAL EQUATION

Let $\Omega = C([e,g],\mathbb{R})$ be the set of all the continuous functions with the domain of real values and defined on $[e,g]$.

Now, we let the integral equation

$$\varpi(l) = f(j) + \delta \int_e^g F(l,j)\varpi(l)\sigma j \text{ for } l,j \in [e,g], \tag{3.1}$$

where $\delta > 0$, $f(j)$ is a function of $j: j \in [e,g]$ and $F \in \Omega$. Define $\mathcal{M}, \mathcal{N}$, and $\mathcal{O}$ by

$\mathcal{M}(\varpi(l),\omega(l),\tau) = \sup_{l\in[e,g]} \frac{\tau}{\tau+|\varpi(l)-\omega(l)|}$ for all $\varpi,\omega \in \Omega$ and $\tau > 0$,

$\mathcal{N}(\varpi(l),\omega(l),\tau) = 1 - \sup_{l\in[e,g]} \frac{\tau}{\tau+|\varpi(l)-\omega(l)|}$ for all $\varpi,\omega \in \Omega$ and $\tau > 0$,

and

$\mathcal{O}(\varpi(l),\omega(l),\tau) = \sup_{l\in[e,g]} \frac{|\varpi(l)-\omega(l)|}{\tau}$ for all $\varpi,\omega \in \Omega$ and $\tau > 0$,

with CTN and CTCN defined by $\pi * \mu = \pi\mu$ and $\pi \lozenge \mu = \max\{\pi,\mu\}$. Then, $(\Omega,\mathcal{M},\mathcal{N},\mathcal{O},*,\lozenge)$ be a complete NMS. Assuming that

$|F(l,j)\varpi(l) - F(l,j)\omega(l)| \leq |\varpi(l) - \omega(l)|$

for $\varpi,\omega \in \Omega$, $\Phi \in (0,1)$ and $\forall\, l,j \in [e,g]$. Also consider $\left(\delta \int_e^g \sigma j\right) \leq \Phi < 1$. Then, the integral equation in equation (3.1) has a unique solution.

Proof: Define $\zeta\colon \Omega \to \Omega$ by

$\zeta\varpi(l) = f(j) + \delta \int_e^g F(l,j)e(l)\sigma j$ for all $l,j \in [e,g]$.

The existence of a FP of the operator ζ is equal to the existence of the solution of an integral equation.

Now, for all $\varpi,\omega \in \Omega$, we obtain

$$\mathcal{M}(\zeta\varpi(l),\zeta\omega(l),\Phi(\tau)) = \sup_{l\in[e,g]} \frac{\Phi(\tau)}{\Phi(\tau) + |\zeta\varpi(l) - \zeta\omega(l)|}$$

$$= \sup_{l\in[e,g]} \frac{\Phi(\tau)}{\Phi(\tau) + \left|f(j) + \delta \int_e^g F(l,j)e(l)\sigma j - f(j) - \delta \int_e^g F(l,j)e(l)\sigma j\right|}$$

$$= \sup_{l\in[e,g]} \frac{\Phi(\tau)}{\Phi(\tau) + \left|\delta \int_e^g F(l,j)e(l)\sigma j - \delta \int_e^g F(l,j)e(l)\sigma j\right|}$$

$$= \sup_{l\in[e,g]} \frac{\Phi(\tau)}{\Phi(\tau) + |F(l,j)\varpi(l) - F(l,j)\omega(l)|\left(\delta \int_e^g \sigma j\right)}$$

$$\geq \sup_{l\in[e,g]} \frac{\tau}{\tau + |\varpi(l) - \omega(l)|}$$

$$\geq \mathcal{M}(\varpi(l),\omega(l),\tau),$$

$$\mathcal{N}(\zeta\varpi(l),\zeta\omega(l),\Phi\,\tau)=1-\sup_{l\in[e,g]}\frac{\Phi(\tau)}{\Phi(\tau)+|\zeta\varpi(l)-\zeta\omega(l)|}$$

$$=1-\sup_{l\in[e,g]}\frac{\Phi(\tau)}{\Phi(\tau)+\left|f(j)+\delta\int_e^g F(l,j)e(l)\sigma j-f(j)-\delta\int_e^g F(l,j)e(l)\sigma j\right|}$$

$$=1-\sup_{l\in[e,g]}\frac{\Phi(\tau)}{\Phi(\tau)+\left|\delta\int_e^g F(l,j)e(l)\sigma j-\delta\int_e^g F(l,j)e(l)\sigma j\right|}$$

$$=1-\sup_{l\in[e,g]}\frac{\Phi(\tau)}{\Phi(\tau)+|F(l,j)\varpi(l)-F(l,j)\omega(l)|\left(\delta\int_e^g \sigma j\right)}$$

$$\leq 1-\sup_{l\in[e,g]}\frac{\tau}{\tau+|\varpi(l)-\omega(l)|}$$

$$\leq \mathcal{N}(\varpi(l),\omega(l),\tau),$$

and

$$\mathcal{O}(\zeta\varpi(l),\zeta\omega(l),\Phi\,\tau)=\sup_{l\in[e,g]}\frac{|\zeta\varpi(l)-\zeta\omega(l)|}{\Phi(\tau)}$$

$$=\sup_{l\in[e,g]}\frac{\left|f(j)+\delta\int_e^g F(l,j)e(l)\sigma j-f(j)-\delta\int_e^g F(l,j)e(l)\sigma j\right|}{\Phi(\tau)}$$

$$=\sup_{l\in[e,g]}\frac{\left|\delta\int_e^g F(l,j)e(l)\sigma j-\delta\int_e^g F(l,j)e(l)\sigma j\right|}{\Phi(\tau)}$$

$$=\sup_{l\in[e,g]}\frac{|F(l,j)\varpi(l)-F(l,j)\omega(l)|\left(\delta\int_e^g \sigma j\right)}{\Phi(\tau)}$$

$$\leq \sup_{l\in[e,g]}\frac{|\varpi(l)-\omega(l)|}{\tau}$$

$$\leq \mathcal{O}(\varpi(l),\omega(l),\tau).$$

Therefore, all circumstances of Corollary 2.1 are fulfilled. Hence, operator ζ has a single FP. This implies that integral **Eq.** (**3.1**) has a unique solution.

4. CONCLUSION

In this research paper, some common fixed-point (CFP) theorems were discussed and proved in the context of G-complete neutrosophic metric space using the alternating distance function (ADF) and defined neutrosophic (Φ , ψ)-weak contraction. This work can be extended further in the context of neutrosophic b-metric spaces and neutrosophic b-metric like spaces.

Conflict of Interest Statement

There is no conflict of interest among author of this article.

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