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Title:	**Existence and Convergence of Fixed Points of Generalized α-Non-Expensive Mappings in Metric Spaces**
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Existence and Convergence of Fixed Points of Generalized α-Non-Expensive Mappings in Metric Spaces

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Abstract

The current study aims to introduce condition (C)on mappings. This condition is median of non-expansive mappings and quasi non-expansive mappings. This study presented some fixed point results of generalized α-non-expansive mappings in metric spaces. Moreover, this study also focuses on the present existence and convergence of prudent results satisfying the condition (C).

Keywords: convergence, condition (*C*), fixed point, non-expansive mapping, opial property, α-non-expansive

INTRODUCTION

Fixed point theory is the most important and essential tool in mathematics for solving different problems. It can be considered as a core subject of nonlinear analysis [1, 2]. In the last few decades, fixed point theory has been a flourishing area of research for the mathematicians [3]. Fixed point of non-expansive mappings was investigated by Browder [4] in the seventh decade of the twentieth century. Later, Godhe [5] and Kirk [6, 7] were the two researchers who did a substantial work in the field of fixed point theory concerning various problems. Whereas in order to obtain the same results Browder [8] and Gohde also discussed a special case of uniformly convex Banach space. The non-expansive mappings are such mappings in which the Lipschitz constant is unity [9]. Suzuki [10] was the first mathematician who introduced some special kinds of generalizations of non-expansive mappings [11]. He also showed that non-expansive mappings of all types except α-non-expansive mappings are continuous on their domains [12, 13]. In 2007, he proved [14] some existence and convergence results in Banach spaces. The current study aims to proceed with this work in metric spaces.

In 2017, Rahul Shukla, Rajendra Pant, and Manuel De La Sen [15] proved that all monotone mappings satisfy condition (C) but the converse is not true. Mathematicians have done a huge work in the field of non-expansive mappings [16] on existence and convergence, and on some important spaces including metric spaces. Metric is basically a function that defines the concept of distance between any two points [17]. Metric space is an advanced and modern field of mathematical sciences like many other fields for instance biological sciences, physical sciences, chemical sciences, commerce, and business have their applications of metric spaces. A metric induces a topology on a set but not all topologies can be generated by metric [18].

This article aims to present some fixed points results in metric spaces and some important existence and convergence results of generalized α-non-expansive mappings in metric spaces. Continuing the results by Tomonari Suzuki in Normed spaces [14], this study presented the results for metric spaces. Some fixed point theorems and their existences and convergences are also elaborated with certain supporting examples.

2. PRELIMINARIES

This section presents some preliminaries. Throughout this article the researcher has denoted $\mathbb{N}$ by the set of all positive integers and $\mathbb{R}$ by the set of real numbers. Let K be a subset of X and $T : K \to K$ be a self-mapping. A point $u \in X$ with $T(u) = u$ is known as a fixed point [19]. Let (X, d) be a metric space. The mapping $T : K \to K$ is called non-expansive [20] if

$$d\big(T(e), T(f)\big) \leq d(e, f)$$

for all $e, f \in K$. If $F(T) \neq \emptyset$ such that

$$d(T(e), f) \leq d(e, f)$$

for all $e \in K$ and $f \in F(T)$ (where $F(T)$ is the set of fixed points T), then T is called a quasi non-expansive mapping. Let K be a non-empty subset of the metric space X. A mapping $T : K \to K$ is said to be α−non-expansive [21], if $\forall\, e, f \in K$ and $0 < \alpha < 1$,

$$d^2(T(e), T(f)) \leq \alpha\, d^2(T(e), f) + \alpha\, d^2(e, T(f)) + (1 - 2\alpha)\, d^2(e, f).$$

A subset L of a real metric space X is known as a closed convex cone [22] if the following conditions hold,

(i) L is non-empty, closed, and $L \neq \{0\}$

(ii) $ae + bf \in L \ \forall \ e, f \ \in L$ and $a, b \ \in R$ with $a, b \ \geq 0$

(iii) $e \ \in L$ and $-e \ \in L$ implies $e = 0$.

Let X be a non-empty set and τ be a collection of subsets of X such that

(i) $\kappa, X \ \in \ \tau$

(ii) Arbitrary union of members of τ is in τ

(iii) Finite intersection of members of τ also belongs to τ.

Then τ is called a topology on X and (X, τ) is called a topological space [23]. A metric space (X, d) is known as uniformly convex in every direction ($UCED$) if for $\epsilon \ \in \ (0, 2]$, $u \ \in \ X$, $d\,(u) = 1, \exists \ \xi\,(\epsilon, u) > 0$ such that

$$d\left(\frac{e+f}{2}\right) \leq 1 - \xi(\epsilon, u);$$

$$\forall \ e, f \ \in \ X, d\,(e) \leq \ 1, d\,(f\,) \ \leq \ 1,$$

$$d\,(e, f\,) \ \in \ \{tu \mid t \ \in \ [-2, -\epsilon] \ \cup \ [\epsilon, 2]$$

X is known as uniformly convex if X is $(UCED)$ and

$$inf\{\xi\,(\epsilon, u) : \ d\,(u) \ = \ 1\} \ > \ 0.$$

$(UCED)$ class is larger than a uniformly convex space. A metric space (X, d) is said to have Opial property [24] if every weakly convergent sequence $\{e_n\}$ in X with weak limit u,

$$\lim_{n\to\infty} inf\ d\,(e_n, u) \ < \ \lim_{n\to\infty} inf\ d(e_n, f\,)$$

$\forall \ f \ \in \ X$ and $f \ \neq \ u$.

A mapping $T : \ K \ \to \ K$ is said to satisfy the condition (C) if $\forall \ e, f \ \in \ K$,

$$\frac{1}{2}\, d\big(e, \ T(e)\big) \leq d(e, f)$$

implies

$$d\ (T\ (e)\ ,T\ (f\ ))\ \leq\ d\ (e,f\ )\ .$$

This technique is median of non-expansive and quasi non-expansive. This paper illustrates some existence and convergence results in metric spaces by using the condition (C).

3. BASIC PROPERTIES

Theorem 3.1. [14] Define a non-increasing function θ from $[0,1)$ onto $(0.5, 1]$ by

$$\theta(l) = \begin{cases} 1 & if\ \ 0 \leq l \leq 0.62 \\ (1-l)l^{-2} & if\ 0.62 \leq l \leq 0.70 \\ (1+l)^{-1} & if\ 0.70\ \leq l < 1 \end{cases}$$

So, in the metric space (X, d), the following are equivalent:

(i) X is complete.

(ii) There exist $l\ \in (0, 1)$ such that every mapping T on X satisfying the following has a fixed point:

$$\theta\ (l)\ d(m,Tm)\ \leq\ d(m,n)$$

implies

$$d(Tm,Tn) \leq\ l\ d(m,n)$$

for all $m,n\ \in\ X$. Theorem 3.1 is meaningful because contraction do not characterize the metric completeness, while Caristi and Kannan mappings do (see [1, 2, 25] also [26]. Since $\lim\limits_{l\to 1}\theta\ (l)\ =\ 0.5$, it is very natural to consider the condition (C) .

Lemma 3.1. [18] Let $\{p_n\}$ and $\{s_n\}$ be bounded sequences in a convex metric space $\{X, d\}$ and let $\eta\ \in (0, 1)$.

Suppose that

$$p_{n+1} = W(s_n, p_n, \eta)$$

and

$$d(s_n, p_n) \leq d(p_{n+1}, p_n)$$

for all $n\ \in\ N$. Then

$$\lim_{n\to\infty} d(s_n, p_n) = 0$$

Proof: Consider,

$$d(s_n, p_{n+1}) = d(s_n, W(s_n, p_n, \eta))$$
$$\leq \eta\, d(s_n, s_n) + (1-\eta) d(s_n, p_n)$$
$$= (1-\eta) d(s_n, p_n)$$
$$= (1-\eta) d(s_n, W(s_{n-1}, p_{n-1}, \eta))$$
$$\leq (1-\eta)[\eta\, d(s_n, s_{n-1}) + (1-\eta) d(s_n, p_{n-1})]$$
$$= (1-\eta)\eta\, d(s_n, s_{n-1}) + (1-\eta)^2 d(s_n, p_{n-1})$$
$$= (1-\eta)\eta\, d(s_n, s_{n-1}) + (1-\eta)^2 d(s_n, W(s_{n-2}, p_{n-2}, \eta))$$
$$\leq (1-\eta)\eta\, d(s_n, s_{n-1}) + (1-\eta)^2[\eta\, d(s_n, s_{n-2}) + (1-\eta) d(s_n, p_{n-2})]$$
$$= (1-\eta)\eta\, d(s_n, s_{n-1}) + (1-\eta)^2 \eta\, d(s_n, s_{n-2}) + (1-\eta)^3 d(s_n, s_{n-3})]$$
$$= \lim_{n\to\infty}[\,(1-\eta)\eta\, d(s_n, s_{n-1}) + (1-\eta)^2 \eta\, d(s_n, s_{n-2})]$$
$$+(1-\eta)^3 d(s_n, s_{n-3}) + \ldots + (1-\eta)^n d(s_n, p_1) = 0$$

Thus, we have obtained the required result.

Now we discuss basic properties on condition (C) . The following propositions are obvious.

Proposition 3.1. Every non-expansive mapping satisfies the condition $(\boldsymbol{C})$.

Proof: By definition of non-expansive mappings

$$d\,(Tp, Tq) \leq d\,(p, q)$$

which proves this result.

This statement is naturally true.

Proposition 3.2. Suppose that a mapping $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$ and has a fixed point $\boldsymbol{l} \in \boldsymbol{K}$, with

$$T\,(l) = l$$

Then T is a quasi non-expansive mapping.

Proof: Since

$$T\,(l) = l = d\,(l, Tl)$$

$= 0.5d(l, Tl) \leq d(l, m)$

where $m \in K$ and $l \in F(T)$. By condition (C)

$d(Tl, Tm) \leq d(l, m)$

$\Rightarrow d(l, Tm) \leq d(l, m)$.

So, T is a quasi non-expansive mapping.

Example 3.1. Define a mapping $\boldsymbol{T}$ on $[\mathbf{0}, \mathbf{3}]$ by $\boldsymbol{Tp} = \left\{ \begin{matrix} \mathbf{0} \;\; \boldsymbol{if} \; \boldsymbol{p} \neq \mathbf{3} \\ \mathbf{1} \;\; \boldsymbol{if} \; \boldsymbol{p} = \mathbf{3} \end{matrix} \right\}$

Then, $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$; but $\boldsymbol{T}$ is not non-expensive.

Proof: If $\boldsymbol{p} < \boldsymbol{s}$ and $(\boldsymbol{p}, \boldsymbol{s}) \in \frac{([\mathbf{0},\mathbf{3}]\times[\mathbf{0},\mathbf{3}])}{((\mathbf{2},\mathbf{3})\times\{\mathbf{3}\})}$, so $\boldsymbol{d}(\boldsymbol{Tp}, \boldsymbol{Ts}) \leq \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{s})$ holds. If $\boldsymbol{p} \in (\mathbf{2}, \mathbf{3})$ and $\boldsymbol{s} \in \mathbf{3}$, then $\mathbf{0.5}\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{Tp}) = \mathbf{0.5}\boldsymbol{p} > \mathbf{1} > \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{s})$ and $\mathbf{0.5}\boldsymbol{d}(\boldsymbol{s}, \boldsymbol{Ts}) = \mathbf{1} > \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{s})$ hold. So, $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$

However, T is discontinuous and T is not non-expansive.

Example 3.2. Define a mapping $\boldsymbol{T}$ on $[\mathbf{0}, \mathbf{3}]$ by $\boldsymbol{Tp} = \left\{ \begin{matrix} \mathbf{0} \;\; \boldsymbol{if} \; \boldsymbol{p} \neq \mathbf{3} \\ \mathbf{2} \;\; \boldsymbol{if} \; \boldsymbol{p} = \mathbf{3} \end{matrix} \right\}$.

Then $\boldsymbol{F}(\boldsymbol{T}) \neq \emptyset$ and $\boldsymbol{T}$ is quasi non-expensive, but $\boldsymbol{T}$ does not satisfy condition $(\boldsymbol{C})$.

Proof: It is obvious that $\boldsymbol{F}(\boldsymbol{T}) = \{\mathbf{0}\} \neq \emptyset$ and $\boldsymbol{T}$ is quasi non-expansive. However, since

$\mathbf{0.5}\,\boldsymbol{d}(\mathbf{3}, \boldsymbol{T}\mathbf{3}) = \mathbf{0.5} \leq \mathbf{1} = \boldsymbol{d}(\mathbf{3}, \mathbf{2})$

and

$\boldsymbol{d}(\boldsymbol{T}\mathbf{3}, \boldsymbol{T}\mathbf{2}) = \mathbf{2} > \mathbf{1} = \boldsymbol{d}(\mathbf{3}, \mathbf{2})$

hold, $\boldsymbol{T}$ does not satisfy the condition $(\boldsymbol{C})$.

Lemma 3.2. Suppose that $\boldsymbol{T}$ is a mapping on a closed subset $\boldsymbol{K}$ of a convex metric space $\boldsymbol{X}$. Assume that $\boldsymbol{T}$ satisfies monotone generalized α-non-expansive mappings, $\boldsymbol{F}(\boldsymbol{T})$ is closed, $\boldsymbol{X}$ is strictly convex and $\boldsymbol{K}$ is convex. Then, $\boldsymbol{F}(\boldsymbol{T})$ is convex.

Proof: Let $\{\boldsymbol{w_n}\}$ be a sequence in $\boldsymbol{F}(\boldsymbol{T})$ and converges to some point $\boldsymbol{w} \in \boldsymbol{K}$. Since

$0.5d(w_n, Tw_n) = 0$

$\leq \boldsymbol{d}(\boldsymbol{w_n}, \boldsymbol{w})$

for $n \in N$ we obtain

$$\lim_{n \to \infty} \sup d\,(w_n, Tw)$$
$$= \lim_{n \to \infty} \sup d\,(Tw_n, Tw)$$
$$\leq \lim_{n \to \infty} \sup d\,(w_n, w)$$
$$= 0$$

That is $\{w_n\}$ converges to $T(w)$. This implies that $T(w) = w$.

Therefore, $F(T)$ is closed. We suppose that X is strictly convex and K is convex. We fix $\zeta \in (0,1)$ and $p, s \in F(T)$ with $p \neq s$, and put $w = W(p, s, \zeta) \in K$. Then we have

$$d\,(p,s) \leq d\,(p,Tw) + d\,(s,Tw)$$
$$= d\,(Tp,Tw) + d\,(Ts,Tw)$$
$$\leq d\,(p,w) + d\,(s,w)$$
$$= d\,(p,s).$$

From the strict convexity of X, there exist $\xi \in [0,1]$ such that

$$T\,w = W\,(p,s,\xi)\,.$$

Since

$$(1 - \xi)\,d\,(p,s)$$
$$= d\,(Tp,Tw) \leq d\,(s,w)$$
$$= (1 - \zeta)\,d\,(p,s)$$
$$= \xi d\,(p,s)$$
$$= d\,(Tp,Tw)$$
$$\leq d\,(p,w)$$
$$= \zeta d\,(p,s)\,,$$

we get $1 - \xi \leq 1 - \zeta$ and $\xi \leq \zeta \Leftrightarrow \xi = \zeta$. Therefore we have obtained $w \in F(T)$.

Lemma 3.3. [6] Let T be a mapping on a subset K of a metric space X. Assume that T satisfies the condition (C).

Then for all $e, f \in K$, the following holds:

(i) $d(Te, T^2e) \leq d(e, Te)$.

Either

(ii) $0.5d(e, Te) \leq d(e, f)$

or

$0.5d(Te, T^2e) \leq d(Te, Tf)$.

Again either

(iii) $d(Te, Tf) \leq d(e, f)$

or

$d(T^2e, Tf) \leq d(Te, f)$.

Proof: (i) follows from

$0.5\, d(e, Te) \leq d(e, Te)$.

(iii) follows from (ii). Let us prove (ii). Arguing by contradiction, we assume that

$0.5\, d(e, Te) > d(e, f)$

and

$0.5d(Te, T^2e) > d(Te, f)$.

Then we have by (i)

$d(e, Te) \leq d(e, f) + d(Te, f)$.

$< 0.5d(e, Te) + 0.5d(Te, T^2e)$

$\leq d(e, Te)$.

This is a contradiction. So, we obtain the required result.

4. CONVERGENCE RESULTS

In this section, we give two convergence theorems for mappings with the condition (C). First, we will prove some important lemmas which will play important roles in this article.

Lemma 4.1. Let T be a mapping on a bounded convex subset K of a metric space X. Let T satisfy condition the (C).

Consider the sequence $\{p_n\}$ in K by $e_1 \in K$ and

$p_{n+1} = W(Tp_n,\ p_n, \xi),$

for $n \in N$, where ξ belongs to real numbers and $\xi\ \in\ [0.5, 1)$ then

$\lim_{n\to\infty} d(Tp_n,\ p_n) = 0$

Proof: Consider $\xi\ \in\ [0.5, 1),$ then we have

$0.5d(p_n, Tp_n)$

$\leq \xi d(p_n, Tp_n)$

$= d(p_n, w(Tp_n, p_n, \xi)\ ),$

for $n \in \mathbb{N}$. Hence

$d(p_n, w(Tp_n, p_n, \xi)\ ) \leq \xi d(p_n, Tp_n)$

Also by condition the (C), we have

$d(Tp_n, w(Tp_n, p_n, \xi)\ ) \leq d(p_n, w(Tp_n, p_n, \xi)).$

By using Lemma 3.1, we have

$\lim_{n\to\infty} d(Tp_n,\ p_n) = 0.$

Lemma 4.2. Let T be a mapping on a subset K of a metric space X. Assume that T satisfies the condition (C). Then

$d\ (p, Ts)\ \leq\ 3d\ (Tp, p)\ +\ d\ (p, s)$

holds for all $p, s\ \in\ K.$

Proof: By Lemma 3.3 case (*iii*), we have

$d\ (Tp, Ts)\ \leq d\ (p, s)$

or

$d\ (T^2p, Ts) \leq\ d\ (Tp, s)$

holds. In the first case, we have

$d\ (p, Ts)\ \leq d\ (p, Tp)\ +\ d\ (p, s)$

$\Rightarrow d\ (p, Ts) \leq d\ (p, Tp) +\ d\ (Tp, T^2p) + d(T^2p, Ts)$

$\leq d\ (p, Tp) + d\ (p, Tp) + d(Tp, s)$

$\leq 2d\,(p, Tp) + d\,(Tp, p) + d(p, s)$

$= 3d\,(p, Tp) + d(p, s).$

Thus we have obtained the desired results in both cases.

Using the above two lemmas, we can prove the following, which is connected with Ishikawa's convergence theorems.

Theorem 4.1. Let $\boldsymbol{T}$ be a mapping on a compact convex subset $\boldsymbol{K}$ of a metric space $\boldsymbol{X}$. Assume that $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$. Define a sequence $\{\boldsymbol{p_n}\}$ in $\boldsymbol{K}$ by $\boldsymbol{p_1} \in \boldsymbol{K}$ and

$$p_{n+1} = W(Tp_n, p_n, \zeta)$$

for $n \in N$, where $\zeta \in [0.5, 1)$. Then $\{p_n\}$ converges strongly to a fixed point of T.

Proof: By Lemma 4.1, we have

$$\lim_{n\to\infty} d(Tp_n, p_n) = 0$$

Since $\boldsymbol{K}$ is compact, there exist subsequence $\{\boldsymbol{p_{n_j}}\}$ of $\{\boldsymbol{p_n}\}$ and $\boldsymbol{w} \in \boldsymbol{K}$ such that $\boldsymbol{p_{n_j}}$converges to $\boldsymbol{w}$. By Lemma 4.2, we get

$$d\left(p_{n_j}, Tw\right) \leq 3d\left(Tp_{n_j}, p_{n_j}\right) + d\left(p_{n_j}, w\right)$$

for all $\boldsymbol{j} \in \boldsymbol{N}$. Therefore $\{\boldsymbol{p_{n_j}}\}$ converges to $\boldsymbol{Tw}$. This implies that $\boldsymbol{Tw = w}$.

That is, $\boldsymbol{w}$ is a fixed point of $\boldsymbol{T}$. From Proposition 3.2, we have

$$d(p_{n+1}, w) = d(W(Tp_n, p_n, \zeta), w)$$

$$\leq \zeta d(Tp_n, w) + (1-\zeta)d(p_n, w)$$

$$\leq \zeta d(p_n, w) + (1-\zeta)d(p_n, w)$$

$$= d(p_n, w)$$

for $\boldsymbol{n} \in \boldsymbol{N}$. So $\{\boldsymbol{p_n}\}$ converges to $\boldsymbol{w}$.

Next we have proved a convergence theorem connected with Edelstein and O'Briens's. Before this, we have given the following propositions:

Proposition 4.1. Let $\boldsymbol{T}$ be a mapping on a subset $\boldsymbol{K}$ of a metric space $\boldsymbol{X}$ with the Opial property. Consider $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$. If $\{\boldsymbol{u_n}\}$ converges weakly to $\boldsymbol{w}$ and

$$\lim_{n\to\infty} d(Tp_n, p_n) = 0,$$

then

$$T(w) = w$$

That is,

$$(I - T)w = 0.$$

Proof: By Lemma 4.2, we get

$$d(p_n, Tw) \le 3d(Tp_n, p_n) + d(p_n, w) \qquad (4.1)$$

for $n \in N$ and hence it is given that

$$\lim_{n\to\infty} d(Tp_n, p_n) = 0.$$

Putting in **Eq.** (**4.1**), we get

$$d(p_n, Tw) \le d(p_n, w)$$

$$0 \le \liminf_{n\to\infty} d(p_n, Tw) \le \liminf_{n\to\infty} d(p_n, w)$$

By the Opial property, we obtain

$$Tw = w.$$

Theorem 4.2. Let T be a mapping on a weakly compact convex of subset K of a metric space X with the Opial property. Suppose T satisfies the condition (C). Consider a sequence $\{p_n\}$ in K by $p_1 \in K$ and

$$p_{n+1} = W(Tp_n, p_n, \zeta)$$

for $n \in N$, where $\zeta \in \mathbb{R}$ and $\zeta \in [0.5, 1)$. Then $\{p_n\}$ converges weakly to a fixed point of T.

Proof: By Lemma 4.1, we have

$$\lim_{n\to\infty} d(Tp_n, p_n) = 0$$

Since K is weakly compact and there a exist subsequence $\{p_{n_j}\}$ of $\{p_n\}$ as well as $w \in K$ such that $\{p_{n_j}\} \to w$. By proposition 4.1, we get $T(w) = w$. By the result of Theorem 4.1, we can conclude $\{d(p_n, w)\}$ is a non-increasing sequence. On the contrary, we suppose that $\{p_n\}$ does not converges to $\boldsymbol{w}$. Then there exists a subsequence $\left\{\boldsymbol{p}_{\boldsymbol{n}_j}\right\}$ of $\{\boldsymbol{p}_{\boldsymbol{n}}\}$ as well as $\boldsymbol{w} \in \boldsymbol{K}$ such that $\left\{\boldsymbol{p}_{\boldsymbol{n}_j}\right\} \rightharpoonup \boldsymbol{w}$ and $\boldsymbol{l} \neq \boldsymbol{w}$. We note that $\boldsymbol{T}(\boldsymbol{l}) = \boldsymbol{l}$. By the Opial property,

$$\begin{aligned}
&\lim_{n\to\infty} \inf d(p_n, w) \\
&= \lim_{n\to\infty} \inf d\left(p_{n_j}, w\right) \\
&< \lim_{n\to\infty} \inf d\left(p_{n_j}, l\right) \\
&= \lim_{n\to\infty} \inf d(p_n, l) \\
&= \lim_{n\to\infty} \inf d\left(p_{n_k}, l\right) \\
&< \lim_{n\to\infty} \inf d\left(p_{n_k}, w\right) \\
&= \lim_{n\to\infty} \inf d(p_n, w)
\end{aligned}$$

This is a contradiction. Thus, we have obtained the required result.

5. EXISTENCE RESULTS

In this section, we prove existence theorems of fixed points of mappings with the condition (C) .

Theorem 5.1. Let $\boldsymbol{T}$ be a mapping on a convex subset $\boldsymbol{K}$ of a metric space $\boldsymbol{X}$. Suppose that $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$. Suppose further that either of the following holds:

(i) K is compact;

(ii) K is weakly compact and X has the Opial property. Then T has a fixed point.

Proof:

(i) If K is compact, then from theorem 4.1, we have a sequence $\{p_n\}$ converging to a fixed point p of T.

(ii) If K is weakly compact and having the Opial property. Then from theorem 4.2, we get a subsequence $\{p_n\}$ converging to a fixed point p of T.

Theorem 5.2. Let $\boldsymbol{K}$ be a weakly compact convex subset of a uniformly convex in every direction metric space $\boldsymbol{X}$. Let $\boldsymbol{T}$ be mapping on $\boldsymbol{K}$. Assume that $\boldsymbol{T}$ satisfies the condition $(\boldsymbol{C})$. Then $\boldsymbol{T}$ has a fixed point.

Proof: Define a sequence $\{p_n\}$ in K with

$$p_{n+1} = W(Tp_n\,, p_n, 0.5)$$

for $n \in N$. Then by lemma 4.1, we have

$$\lim_{n\to\infty} \sup d(Tp_n,\ p_n) = 0$$

holds. Now, we define a continuous convex function $f : k \to (0,\infty)$ by

$$f(p) = \lim_{n\to\infty} \sup d\ (Tp_n,\ \ p) = 0$$

$\forall\, p \in k.$

Since K is weakly compact and f will be weakly lower semi continuous, there exists $w \in K$ such that

$f\ (w) = min\{f\ \ (p) : p \in K\}$. By lemma 4.2, we get

$d\ (p_n, Tw) \leq 3d\ \ (Tp_n,\ p_n) + d\ (Tp_n,\ \ w).$

By taking *lim* sup on both sides and using lemma 4.1 we get $f\ (Tw) \leq f\ (w)$. Moreover, $f\ (w)$ is minimum. So, $f\ (Tw) = f\ (w)$. Next, we show that $f\ (w) = w$. On the contrary assume that $T\ (w) \neq w$. Then by using the strict semi continuity of *f* we have

$$f\ (w) \leq f\ (W\ \ (Tw, w, 0.5)) < max\{f\ (w)\,, f\ \ (Tw)\} = f\ (w)\,.$$

This is a contradiction. Hence $T\ (w) = w$.

Theorem 5.3. Let K be a weakly compact convex subset of a uniformly convex in every direction metric space X. Let Q be a family of commuting mappings on K satisfying the condition (C) . Then Q has a common fixed point.

Proof: Let $\boldsymbol{T_1}, \boldsymbol{T_2}, \boldsymbol{T_3} \ldots . \boldsymbol{T_m} \in \boldsymbol{Q}$. By theorem 5.2, $\boldsymbol{T_1}$ has a fixed point in $\boldsymbol{K}$ and

$$F\ \ (T_1) \neq \emptyset.$$

By lemma 3.2, $F\ (T_1)$ is a closed and convex subset. We also assumed that

$$B := \bigcap_{i=1}^{j-1} F\ (T_i)$$

in non-empty, closed and convex for some $j\ \in\ N$ with $1\ <\ j\ \leq\ m$. For $p\ \in\ B$ and $i\ \in\ N$ with $1\ \leq\ i\ <\ j$ and

$$T_j\ \circ T_i\ =\ T_i\ \circ T_j,$$

we have

$$T_j p\ =\ T_j\ \circ T_i\ p\ =\ T_i\ \circ T_j p.$$

Thus, $T_j p$ is a fixed point of T_i, which implies that $T_j p\ \in B$. Therefore, we obtain$T_j(B) \subset B$. By theorem 5.2, we get T_j has a fixed point in B and

$$B\ \cap F\big(T_j\big) = \bigcap_{i=1}^{j} F\ (T_i)\ \neq \emptyset.$$

Moreover, the set is closed and convex, by lemma 3.2. Through the induction process we obtained $\bigcap_{i=1}^{l} F\ (T_i)\ \neq \emptyset$. In other words, $\{F(T\ )\ ,\ T \in Q\}$ has a finite intersection property. Since K is weakly compact and $F(T\ )$ is weakly closed for every $T\ \in\ Q$, we get $\bigcap_{T\in Q} F\ (T)\ \neq \emptyset$.

6. CONCLUSION

The current study discussed some important conditions on different kinds of mappings. These conditions were weaker than the non-expansive mappings and stronger than the quasi-non-expansive mappings. Moreover, this study also presented some fixed point theorems with their existence and convergence results of generalized α-non-expansive mappings in metric spaces by applying the condition (C). The contemporary research presented some propositions, lemmas, and examples for the future illustration.

Conflict of Interest Statement

There is no conflict of interest among author of this article.

Data Availability Statement

The data sets generated during this work are available by the corresponding author on reasonable request.

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