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Monotone Data Modelling Using Rational Cubic Fractal Interpolation Function

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ABSTRACT

Geometric modelling of several intricate and complex structures such as trees, mountains, clouds, ferns, geographic topography, and coastlines is challenging in computer graphics. Traditional splines such as trigonometric, polynomial, exponential, and rational fail to simulate this significant class of complex structures, which are highly irregular in nature. For this purpose, this research develops a novel cutting-edge method for synthesizing and modelling structures. The proposed technique; C^1 fractal interpolation function (FIF) builds an iterated function system (IFS) by integrating fractal calculus and rational cubic polynomial functions. Appropriate conditions on scaling and shape parameters are derived to help maintain the inherited shape qualities of the data. Experiments in numerous scientific domains, such as the pharmaceutical and chemical industries have been presented as an example, to confirm the usefulness of the suggested model. Moreover, the graphic results demonstrated that the developed monotone hybrid model (MHM) offers a heterogeneous method for gathering data with a monotone structure.

Keywords: fractal interpolation, hybrid model, monotone interpolation, spline functions

1. INTRODUCTION

Interpolation is an indispensable tool for estimating the unknown function for data modelling. There are numerous interpolation techniques, which use various function families, including polynomial, exponential, rational, trigonometric, and spline. However, these traditional non-recursive approaches resulted in interpolants, which were differentiable several times, with the possible exception of a limited number of points and restricted model smooth structures. Many signals in the real world and in

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experimental data are complex and their composition frequently demonstrates irregularities and hardly ever appears to be smooth. Thereby, interpolants that could picture non-smooth signals, as well as smooth structures, were highly required. The prime purpose of this research is to provide an innovative framework for the approximate representation of non-smooth data from natural (feathers, trees, leaves, clouds, feathers, flowers, glaciers, landscapes, and galaxies), engineering or scientific phenomena together with the collection of smooth data. A novel form of fractal interpolation function is created utilizing the rational spline containing a cubic polynomial in the numerator as well as the denominator.

One of the pivotal characteristics of data generated from countless physical phenomena, scientific problems, and engineering applications is monotonicity where entities only have a meaning when their values are monotone. For instance, in the field of engineering, Tensile Strength, Ultimate Tensile Strength or Ultimate Strength is not an uncommon concept. It refers to the maximum level of stress that a structure or material can tolerate while being pulled or stretched before getting breaking. This phenomenon gave rise to monotone data. The applied forces, commonly called stress and the stretch, referred to as strain, are always in a monotonic relationship where one varies with the change in the value of the other [1]. Certain devices, such as; digital-to-analog (DAC used in audio video devices) and analog-to-digital (ADC used in music recording and digital signal processing), have monotonicity an essential characteristic used in feedback control loops [1]. In CAD and DAC, the digital output code always increases as the analog voltage input increase and vice versa. Monotonicity can be seen in Newton’s law of cooling in which the rate of heat loss is directly proportional to the differences in temperature between the bodies [2].

Monotonicity also appears in other phenomena, such as the content of atrazine and nitrate in shallow ground waters, the link between the partial pressure of oxygen, and the percentage dissociation of hemoglobin percentage dissociation and oxygen partial pressure, erythrocyte sedimentation rate in patients with cancer and concentration of uric acid level in a patient affected from gout [3]. It is a fundamental function behind the renowned ‘Probability theory’ that uses the monotonically increasing function of a random variable in its cumulative distribution.

Natural phenomena can produce monotonic positive data as well. The relationship between a child's height and their intake of food and energy can be considered as an example. For instance, height is a positive quantity and always goes up. There are several other examples in everyday experiences where objects only create meaning when their values are monotonous. Examples are surfaces and the dose-response curve in biochemistry and pharmacology; consumption function that describes the relationship between a household's consumption and its available income in economics, statistical approximations of copulas and quasi-copulas, [4] and empirical option pricing models in finance [4].

Based on different approaches (such as classical spline, spline with knots, and fractal spline) various monotonicity-preserving models for monotone data have been discussed in the previous literature [1-18]. Spline functions have also been used for the solution of partial differential equations [19, 20]. Previously, spline functions have become the main tools for solving monotonicity problems. For instance, Crisp and Hussain [5] solved the issue of monotonicity through Bernstein-Bezier rational cubic functions. Spline functions were also used to provide multiple kinds of monotonicity methods for bivariate functions on triangles [6].

Using different types of spline functions, some other useful techniques and algorithms were supplied to address the problem of monotonicity [1, 3, 7]. Typically, data that comes from experiments (for instance, information drawn from signals in the actual world, including seismic information, financial series, and bioelectric recordings are complex and immensely irregular. As a result, interpolating these data using traditional interpolation techniques became an inappropriate challenge. Barnsley [8] made an attempt by introducing a new interpolation method called Fractal Interpolation using a special type of iterated function system. It provided a powerful framework to deal with highly irregular data. However, consistently, when it is a prerequisite that the interpolant should reflect the inherited shape features (convexity, positivity, monotonicity) of the irregular data set, fractal interpolation functions failed to secure the shape properties of data. Later, Barnsley and Harrington [9] noticed that in case the problem is of shape preservation type, then the parameters of the IFS may be chosen suitably so that the corresponding FIF preserve the inherited properties of the data. This observation initiated a striking relationship between the classical splines interpolation functions and the fractal interpolation functions.

Recently, with the help of Barnsley and Harrington's results, various spline functions have been generalized in the form of fractal spline functions for instance [4, 10]. Fractal spline [11] though interpolating both regular and irregular data does not guarantee that the model satisfies the monotone trend lies in their data. Non-monotone visual models of the aforementioned experiments misguide us about the actual scenario (which might help scientists to recognize and evaluate various patterns and artifacts in their data) in these experiments. Hence, the visual model must maintain the geometric property of the experimental data in order to achieve precise and proficient results.

The goal of the underlying research is to build a fractal geometry-based function to simulate hidden patterns and future behaviour in many disciplines. A monotone hybrid model (MHM) which is a combination of fractal interpolation function (FIF) and spline interpolation model (SIM) was presented for this purpose. The shape-preserving model is then created with the help of the proposed function to handle the non-linearity problem that the traditional interpolation model was unable to solve. Additionally, the proposed class surpasses its conventional non-recursive counterpart in approximating monotone functions with varied irregularities/fractality in their first derivatives (smooth to nowhere differentiable).

The proposed hybrid model offered a variety of shape control parameters that can be sufficient and highly useful to achieve desired results. To meet the objective, a general piecewise rational cubic fractal interpolation function (GPRC FIF) [11] is used in the construction of MHM.

The proposed model has the following useful features in comparison with the existing models:

- The proposed scheme converts the hybrid model [11] to a monotone hybrid model.
- The proposed hybrid model MHM is the generalization of the model presented in [7].
- Unlike [12], no derivative constraints are imposed in proposed model as MHM works for both data and data with derivatives proficiently.

- Once, the shape parameters are selected, the curve representation becomes unique in its description.
- Existing methods have no degree of freedom [1, 3, 12, 13], whereas the proposed model gives two degrees of freedom.
- The proposed model works for larger as well as smaller data, whereas in [14], constraints are developed on the subinterval lengths for data modelling.

Four shape parameters are used in the proposed model, whereas in [3, 15] one and two shape parameters are used respectively. The remaining sections are structured as follows. Section 2 focuses on a brief introduction to the basic methodology of the hybrid model developed in [11]. In Section 3, a new scheme is developed, which converts the hybrid model to a monotone hybrid model, for solving the problem of monotonicity. Whereas the proposed model is tested on different real data sets to verify their usefulness in Section 4. Comparative analysis has been presented in Section 5 and finally, Section 6 concludes with the significance of the proposed study.

2. METHODOLOGY

In this section, the hybrid model (GPRC FIF) [11] based on the fractal framework and spline function was briefly reviewed. The fractal framework was built on an iterative function system (IFS), which is defined below comprehensively.

2.1. IFS Theory

Let the real interval $I[r_1, r_n]$ be partitioned such that $r_1 < r_2 < \cdots < r_n$. Suppose the data set is given $\{(r_i, \hat{v}_i) \in I \times E : i = 1,2, \dots, n\}$, where E is denoted as a suitable compact set containing all $\hat{v}_i{'s}$. Take $I_i = [r_i, r_{i+1}]$, then the two mappings given below would be contraction homeomorphic so that $\varepsilon_i : I \to I_i$,

with

$$\varepsilon_i(r_1) = r_i, \;\; \varepsilon_i(r_n) = r_{i+1}, \quad i = 1,2, \dots, n-1. \tag{2.1}$$

$$|\varepsilon_i(e_1) - \varepsilon_i(e_2)| \le l_i |e_1 - e_2|, \;\; \forall\, e_1, e_2 \in I, \text{ for some } \quad 0 < l_i < 1.$$

Suppose, $A = I \times E$ and there is a mapping $\sigma_i : A \to E$, which is continuous if

$$\sigma_i(r_1,\hat{v}_1)=\hat{v}_i,\ \sigma_i(r_n,\hat{v}_n)=\hat{v}_{i+1},\ i=1,2,\dots,n-1. \tag{2.2}$$

$|\sigma_i(r,x)-\sigma_i(r,y)|\le \kappa_i|x-y|\ , r\in I,\ \forall\ x,y\ \in E,$ for some $-1<\kappa_i<1$.

Take $\tau_i: A\to A$, which is defined as:

$$\tau_i(r,\hat{v})=\big(\varepsilon_i(r),\sigma_i(r,\hat{v})\big),\forall\ (r,\hat{v})\in A. \tag{2.3}$$

Where

$\varepsilon_i(r)=a_i r+b_i,\ \sigma_i(r,\hat{v})=\kappa_i\hat{v}+g_i(r),$

with $a_i=\frac{r_{i+1}-r_i}{r_n-r_1},\ \ b_i=\frac{r_n r_i-r_1 r_{i+1}}{r_n-r_1}$.

Here, $g_i(r)$ is any suitable continuous function. The set $\{A;\tau_i: i\in (1,2,3,\cdots,n-1)\}$ is termed as an iterative function system.

2.2.Hybrid Model

The hybrid model [11] φ is termed as GPRC FIF corresponding to the iterative function system as:

$$\varphi\big(\varepsilon_i(r)\big)=\kappa_i\varphi(\mathrm{r})+g_i(r), \tag{2.4}$$

with $g_i(r)=\frac{P_i(r\ )}{Q_i(r\ )}=\frac{P_i(\hat{\mu}\ )}{Q_i(\hat{\mu}\ )}$,

where

$P_i(\hat{\mu}\ )=\eta_i(z_i-\kappa_i\hat{v}_1)(1-\hat{\mu}\ )^3+(\eta_i h_i d_i-(r_n-r_1)\eta_i\kappa_i d_1+\delta_i(\hat{v}_i-\kappa_i\hat{v}_1))\hat{\mu}\ (1-\hat{\mu}\ )^2+(-\omega_i h_i d_{i+1}+(r_n-r_1)\omega_i\kappa_i d_n+\rho_i(\hat{v}_{i+1}-\kappa_i\hat{v}_n))\hat{\mu}\ ^2(1-\hat{\mu}\ )+\omega_i(\hat{v}_{i+1}-\kappa_i\hat{v}_n)\hat{\mu}\ ^3,$

$Q_i(\mu\ )=\eta_i(1-\hat{\mu}\ )^3+\delta_i\hat{\mu}\ (1-\hat{\mu}\ )^2+\rho_i\hat{\mu}\ ^2(1-\hat{\mu}\ )+\omega_i\hat{\mu}\ ^3,\ \ \hat{\mu}\ =\frac{r-r_1}{r_n-r_1},\ \ \hat{v}\in[r_1,r_n].$

Here, $\kappa_i<1$ are the scaling factors, d_i is the derivative value at the knot points and $\eta_i,\delta_i,\rho_i\ ,\omega_i$ are the non-negative shape parameters.

3. A MONOTONE HYBRID MODEL

This section sets up a monotone hybrid model (MHM) to solve the monotonicity problem present in the hybrid model as defined in [11]. In setting up the MHM, certain conditions were imposed to shape the parameters and scaling factors, to acquire the desired model.

Let, the monotone data set $\{(r_i, \hat{v}_i) \in I \times E : i = 1,2, \dots, n\}$ be distributed such that $r_1 \le r_2 \le \cdots \le r_n$ with $\hat{v}_{i+1} \ge \hat{v}_i,\ i = 1,2, \dots, n-1$.

Or alternatively,

$$\xi_i = \frac{\hat{v}_{i+1} - \hat{v}_i}{h_i} \ge 0,\ i = 1,2, \dots, n-1. \tag{3.1}$$

For the function that increases monotonically, the necessary constraints were applied on derivative as:

$$\partial_i \ge 0, i = 1,2, \dots, n. \tag{3.2}$$

For the data set increasing monotonically, there are two possibilities.

Case 1: If $\xi_i = 0$ and $\kappa_i = 0$ particularly for some monotone and collinear data set, the function defined in **Eq.** (**2.4**) is constant in each subinterval. Therefore,

$$\partial_i = \partial_{i+1} = 0, i = 1,2, \dots, n.$$

So that

$$\varphi(r) = \hat{v}_i, \quad \forall\, r \in [r_1, r_n], i = 1,2, \dots, n-1.$$

This demonstrates the process by which a hybrid model automatically transforms into a monotonous hybrid model.

Case 2: If $\xi_i > 0$ and $\xi_i - \kappa_i \xi_n > 0 \Leftrightarrow \kappa_i < \frac{\hat{v}_{i+1} - \hat{v}_i}{\hat{v}_n - \hat{v}_1}$, where $\xi_n = \frac{\hat{v}_n - \hat{v}_1}{h_i}$, for this case, we have the following theorem.

Theorem 3.1: Suppose $\{(r_i, \hat{v}_i) : i = 1,2, \cdots, n\}$ be a monotone data set such that $\hat{v}_i > 0$. Assuming $\eta_i > 0$, $\omega_i > 0$ and ∂_i constitute the derivative at the knot points $\hat{v}_i$. Then the corresponding hybrid function GPRC FIF converts to monotone hybrid model (preserve monotonicity) in each interval if the scaling factors and shape parameters satisfy the following condition:

(i) The scaling factors κ_i, $i = 1,2, \dots, n$, are picked as:

$$0 \le \kappa_i < min\{a_i, \kappa_{1i}^*, \kappa_{2i}^*, \kappa_{3i}^*\}$$

(ii) The shape parameters δ_i, $\rho_i, i = 1,2, \dots, n,$ are chosen as:

$$\delta_i > \frac{\eta_i\left(\partial_i - \frac{\kappa_i}{a_i}\partial_1\right)}{(\xi_i - \kappa_i\xi_n)},\ \rho_i > max\left\{\frac{\omega_i\left(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n\right)}{(\xi_i - \kappa_i\xi_n)}, \left(\frac{3\eta_i\omega_i - \delta_i\omega_i\left(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n\right)}{\delta_i(\xi_i - \kappa_i\xi_n) + \eta_i\left(\partial_i - \frac{\kappa_i}{a_i}\partial_1\right)}\right)\right\}.$$

where

$$\kappa_{1i}^* = \frac{a_i\,\partial_{i+1}}{\partial_n}, \quad \kappa_{2i}^* = \frac{a_i\partial_i}{\partial_1}, \kappa_{3i}^* = \frac{\hat{\nu}_{i+1} - \hat{\nu}_i}{\hat{\nu}_n - \hat{\nu}_1}$$

Proof: Take the monotone data set into consideration without losing generality and

$$\varphi\big(\varepsilon_i(r)\big) = \kappa_i\varphi(\mathrm{r}) + g_i(r)$$

It is clear from single variable calculus that the function, $\varphi\big(\varepsilon_i(r)\big)$ is monotone increasing (decreasing) if and only if $\varphi^{(1)} \geq 0$ ($\varphi^{(1)} \leq 0$).

Here, φ is FIF, so derivative $\varphi^{(1)}$ is also FIF and satisfies functional equation. After careful calculation, we arrived at the following equation by differentiating the above equation with respect to r.

$$\varphi\big(\varepsilon_i(r)\big)^{(1)} = \frac{\kappa_i\,\varphi(r)^{(1)}}{a_i} + g_i{}^{(1)}\,(r) \tag{3.3}$$

The hybrid function defined in (4) becomes monotone hybrid model if $\varphi^{(1)} \geq 0$ and only if $g_i{}^{(1)} \geq 0$ for all $r \,\epsilon\, [r_i, r_{i+1}]$.

Taking that $\kappa_i \geq 0\ \forall\ i \in \{1,2,3,\cdots,n\}$, it is evident that $\varphi^{(1)} \geq 0\ ,\forall\ r \in [r_1, r_n]$ if

$$g_i{}^{(1)}\,(r) = \sum_{j=1}^{6} \hat{c}_{ji}\,\frac{\hat{\mu}^{\ 6-j}(1-\hat{\mu}\ )^{j-1}}{[q_i(\hat{\mu}\ )]^2} \geq 0\ ,\forall\ \mu \in [0,1]. \tag{3.4}$$

where

$$\hat{c}_{1i} = \omega_i^2(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n),$$

$$\hat{c}_{2i} = \hat{c}_{1i} + 2\omega_i(\delta_i(\xi_i - \kappa_i\xi_n) - \eta_i(d_i - \frac{\kappa_i}{a_i}\partial_1)),$$

$$\hat{c}_{3i} = (\hat{c}_{2i} - \hat{c}_{1i}) - \big(3\eta_i\omega_i - \delta_i\rho_i(\xi_i - \kappa_i\xi_n)\big) + (\delta_i\omega_i(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n) + \eta_i\rho_i(\partial_i - \frac{\kappa_i}{a_i}\partial_1)),$$

$$\hat{c}_{4i} = (\hat{c}_{5i} - \hat{c}_{6i}) - \big(3\eta_i\omega_i - \delta_i\rho_i(\xi_i - \kappa_i\xi_n)\big) + (\delta_i\omega_i(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n) + \eta_i\rho_i(\partial_i - \frac{\kappa_i}{a_i}\partial_1)),$$

$\hat{c}_{5i} = \hat{c}_{6i} + 2\lambda_i(\rho_i(\xi_i - \kappa_i\xi_n) - \omega_i(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n)),$

$\hat{c}_{6i} = {\eta_i}^2(\partial_i - \frac{\kappa_i}{a_i}\partial_1).$

The denominator in **Eq. 3.4** being a squared quantity is always positive. Furthermore, if $\eta_i \geq 0$ and $\omega_i \geq 0$, the sufficient conditions for monotonicity in each subinterval $[r_i, r_{i+1}]$ are:

$$\hat{c}_{ji} \geq 0, \quad j = 1, 2, \cdots, 6. \quad \hat{c}_{1i} \geq 0 \Leftrightarrow \kappa_i \leq \frac{a_i\partial_{i+1}}{d_n}, \tag{3.5}$$

$$\hat{c}_{6i} \geq 0 \Leftrightarrow \kappa_i \leq \frac{a_i\partial_i}{\partial_1}. \tag{3.6}$$

Also from **Eq.** (**2.1**), it implies that $\varphi \in C^1[r_1, r_n]$,

whenever

$-a_i < \kappa_i < a_i$

and

$$\xi_i - \kappa_i\xi_n > 0 \Leftrightarrow \kappa_i < \frac{\hat{v}_{i+1} - \hat{v}_i}{\hat{v}_n - \hat{v}_1}. \tag{3.7}$$

However, $\hat{c}_{2i} > 0 \Leftrightarrow \delta_i > \frac{\eta_i\left(d_i - \frac{\kappa_i}{a_i}\partial_1\right)}{(\xi_i - \kappa_i\xi_n)}$. (3.8)

$$\text{Similarly, } \hat{c}_{5i} > 0 \Leftrightarrow \rho_i > \frac{\omega_i\left(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n\right)}{(\xi_i - \kappa_i\xi_n)}. \tag{3.9}$$

Finally, both $\hat{c}_{3i} > 0$ and $\hat{c}_{4i} > 0$ iff

$$\rho_i > \left(\frac{3\eta_i\omega_i - \delta_i\omega_i(\partial_{i+1} - \frac{\kappa_i}{a_i}\partial_n)}{\delta_i(\xi_i - \xi_n) + \eta_i(d_i - \frac{\kappa_i}{a_i}\partial_1)}\right), \tag{3.10}$$

provided

$$\delta_i > \frac{\eta_i\left(\partial_i - \frac{\kappa_i}{a_i}\partial_1\right)}{(\xi_i - \kappa_i\xi_n)}.$$

Hence, the results of **Eq. 3.5-3.10**, proved the theorem.

Corollary 1: If $\kappa_i = 0, i = 1,2, \ldots, n-1$ in above **Theorem 3.1**, the sufficient condition convert to

$\delta_i > \frac{\eta_i \partial_i}{\xi_i}$, $\rho_i > max\left\{\frac{\omega_i \partial_{i+1}}{\xi_i}, \left(\frac{3\eta_i\omega_i - \delta_i\omega_i\partial_{i+1}}{\delta_i\xi_i + \eta_i\partial_i}\right)\right\}$.

This is sufficient condition of monotonicity for classical GPRC.

4. RESULTS AND DISCUSSION

This section presents two experiments, which demonstrated the validity and adaptability of the proposed MHM. In these experiments, the derivatives values were calculated through geometric mean method. The monotone data set taken in Table 1 presents the experimental result concerning the watering the great northern beans. Distilled water with pH of 8.5 and Potassium Hydroxide was combined in the chemical solution, which was later used to watering the beans. After 40 days, the beans plant was taken out from vat to weight for the effect of chemical solution. It can be easily noticed that resulting data is monotone. The r-values represented the days and the v-values showed the height of beans.

Table 1. Height of Great Northern Beans

i	1	2	3	4	5	6	7	8
$\boldsymbol{r_i}$	1	2	12	18	24	30	36	40
$\hat{v}_i$	0	0	0.42	2.08	3.43	3.78	4.12	4.37

The data demonstrated in Table 1 is monotone; therefore, the resulting curve must be monotone. However, from Figure 1, which is constructed through hybrid model [11] it can be observed that the arbitrary value (η_i=7, δ_i=4, ρ_i=8.2, ω_i=99, κ_i=0.9) does not preserve the inherited monotonicity of the curve. On the other hand, Figure 2 displays multiple curves, which are produced through the proposed MHM. In Figure 2, it is found that the curve at different iterations preserve the essential characteristic of the data, which is monotonicity, proficiently. Figures 3 and Figure 4 highlighted the effect of diverse values of λ_i and ω_i ,rescpectively. The data presented in these figures showed that with different choices of parameter values through MHM, monotone curve of any particular requirement may be easily gained.

The monotone data set in Table 2 reports the varying degree of saturation of hemoglobin with partial pressure of oxygen. It was observed that saturation of hemoglobin was directly proportional to the partial pressure of oxygen. The r-values represented the partial pressure of oxygen and z-values indicated the percentage of saturation.

Table 2. The varying ability of hemoglobin

$\boldsymbol{i}$	1	2	3	4	5
$\boldsymbol{r_i}$	0	2	8	10	18
$\hat{v}_i$	0	70	91	91	110

The data demonstrated in Table 2 shows the monotone relationship, therefore, the curve must exhibit the same characteristic. Figure 5 is generated through the random values to the shape parameters and scaling factors (η_i=2.9, δ_i=3, ρ_i=7, ω_i=15, κ_i=0.7) in the description of hybrid model as defined in **Eq.** (**2.4**). Figure 5 depicts non-monotone hemoglobin dissociation curve (HDC) which does not make any sense, as the behaviour of the curve should retain monotonicity. This flaw is removed in Figure 6 by using the proposed method presented in Section 3. It can easily be observed that Figure 6 conserve monotonicity at each level of iterations. Figures 6 and 7 showed the profile of HDC with diverse values of η_i and ω_i. It is worth noticing that although various parameters may lead to different results but all conserve monotonicity.

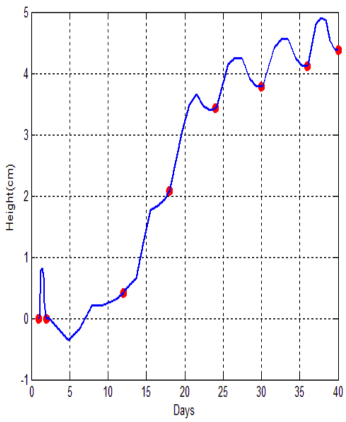

Figure 1. Hybrid model **[1]**

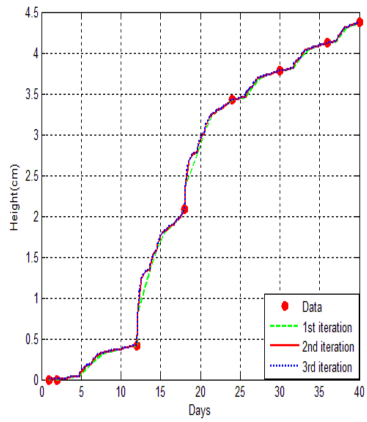

Figure 2. MHM using $\eta_i = 0.5$, $\omega_i = 70$

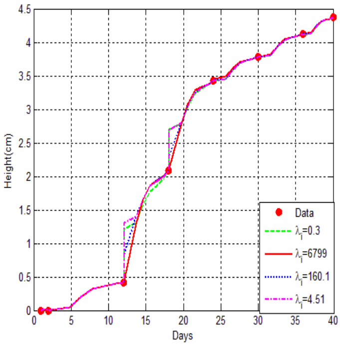

Figure 3. Effect of $\boldsymbol{\eta_i}$ using $\boldsymbol{\omega_i = 9981}$

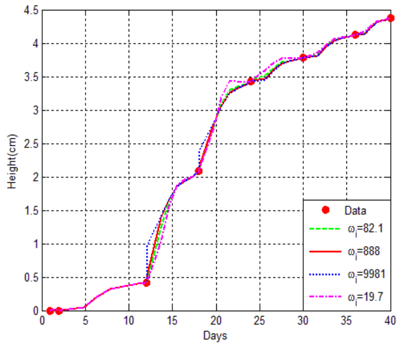

Figure 4. Effect of ω_i using $\eta_i = 87.65$

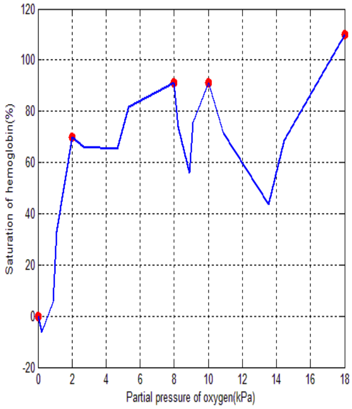

Figure 5. Hybrid model[1]

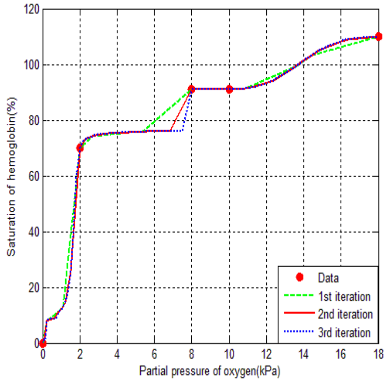

Figure 6. MHM using $\eta_i = 73665$, $\omega_i = 115$

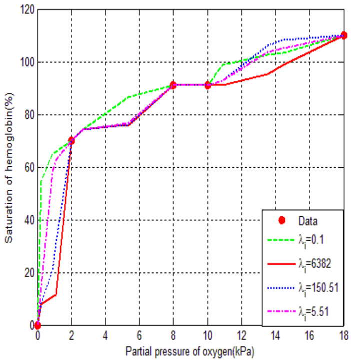

Figure 7. Effect of η_i using $\omega_i = 3.8$

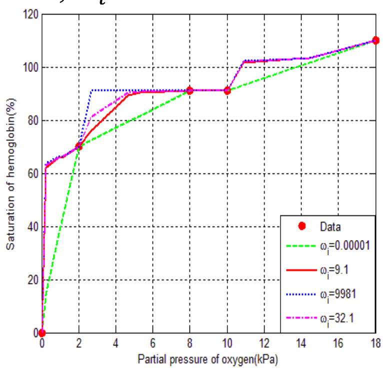

Figure 8. Effect of ω_i using $\eta_i = 0.00001$

4.1. Comparative Analysis

The proposed method guarantees to produce monotone model as compared to the hybrid model [11], which does not ensure to exhibit monotone model when data is monotone. All the existing models [3, 12, 13], failed to provide even a single degree of freedom, so MHM was a noteworthy addition as it offered two degree of freedom to modify and enhance the model. Unlike, [14] and [12], it is easy in the applicability, as it does not constrain the interval length or impose any conditions on derivatives, respectively. The scheme developed in [7] only works for regular data but the proposed approach, which is generalization of [7] is applicable for both regular and irregular datasets.

4.2. Conclusion

General Piecewise Rational Cubic Fractal Interpolation Function (GPRCFIF) was developed to provide a deeper understanding of the data, which displays rough, irregular, and fragmented configuration. Generally, it originates from complex functions or experiments. The GPRCFIF offers the family of four parameters and one scaling factor to modify, upgrade, and change the visual images. One of the most prominent characteristics of the GPRCFIF is its capacity to visualize data with smooth curves by setting all scaling factors to zero. The current study, in particular addressed the problem of monotonicity through the novel proposed model MHM. The experimental results illustrated that the performance of the proposed MHM was very efficient and heterogeneous as it contains the ability (due to recursive nature) to produce multiple monotone curves for the same monotone data according to the user's desire.

REFERENCES

1. Hussain MZ, Hussain M. Visualization of data preserving monotonicity. *Appl Math Comput*. 2007;190(2):1353–1364. https://doi.org/10.1016/j.amc.2007.02.022

2. Sarfraz M, Hussain MZ, Hussain M. Modeling rational spline for visualization of shaped data. *J Numer Math*. 2013;21(1):63–87. https://doi.org/10.1515/jnum-2013-0003

3. Hussain MZ, Sarfraz M, Hussain M. Scientific data visualization with shape preserving C^1 rational cubic interpolation. *Eur J Pure Appl Math*. 2010;2:194–212.

4. Chand AKB, Vijender N, Navescues MA. Shape preserving of scientific data through rational fractal splines. *Calcolo*. 2014;51: 329–362. https://doi.org/10.1007/s10092-013-0088-2

5. Cripps RJ, Hussain MZ. C^1 Monotone cubic Hermite interpolant. *Appl Math Lett*. 2012;25(8):1161–1165. https://doi.org/10.1016/j.aml.2012.02.050

6. Floater MS, Pena JM. Monotonicity preserving on triangles. *Math Comput*. 2000;69(232):1505–1519. https://doi.org/10.1090/S0025-5718-99-01176-X

7. Hussain MZ, Sarfraz M. Monotone piecewise rational cubic interpolation. *Int J Comp Math*. 2009;86(3):423–430. https://doi.org/10.1080/00207160701595145

8. Barnsley MF. Fractal functions and interpolation. *Constr Approx*. 1986;2:303–329. https://doi.org/10.1007/BF01893434

9. Barnsley MF, Harrington AN. The calculus of fractal interpolation functions. *J Approx Theory*. 1989;57:14–34. https://doi.org/10.1016/0021-9045(89)90080-4

10. Balasubramani N. Shape preserving rational cubic fractal interpolation function. *J Comput Appl Math*. 2017;319:277–295. https://doi.org/10.1016/j.cam.2017.01.014

11. Arooj T, Hussain MZ, Hussain F. Scientific Data Visualization via Hybrid Model based on Fractal Spline Interpolation. *Punjab Univ J Math*. 2019;51(9):123–136.

12. Sarfraz M. A rational cubic spline for visualization of monotonic data. *Comput Graph*. 2000;24:509–516. https://doi.org/10.1016/S0097-8493(00)00053-4

13. Sarfraz M. A rational cubic spline for visualization of monotonic data: an alternative approach. *Comput Graph*. 2003;27:107–121. https://doi.org/10.1016/S0097-8493(02)00249-2

14. Lamberti P, Manni C. Shape preserving functional interpolation via parametric cubics. *Numer Algor*. 2001;28:229–254.

15. Sarfraz M, Hussain MZ, Hussain M. Modeling rational spline for visualization of shaped data. *J Numer Math*. 2013;21(1):63–87. https://doi.org/10.1515/jnum-2013-0003

16. Beliakov G. Monotonicity preserving approximation of multivariate scattered data. *BIT Numer Math*. 2005;45(4):653–677. https://doi.org/10.1007/s10543-005-0028-x

17. Sarfraz M, Hussain MZ, Hussain F. Shape preserving curves using quadratic trigonometric splines. *Appl Math Comput*. 2015; 265:1126–1144. https://doi.org/10.1016/j.amc.2015.05.131

18. Hussain MZ, Hussain F, Sarfraz M. Shape preserving positive trigonometry spline curves. *Iran J Sci Technol Trans Sci*. 2018; 42:763–775. https://doi.org/10.1007/s40995-016-0056-1

19. Ahmad B, Perviaz A, Ahmad O, Dayan F. Numerical solution of fourth order homogeneous parabolic partial differential equations using non-polynomial cubic spline method (NPCSM). *Sci Inquiry Rev*. 2021;5(4):19–37. https://doi.org/10.32350/sir/54.02

20. Ahmad B, Perviaz A, Ahmad MO, Dayan F. Solution of parabolic partial differential equations via non-polynomial cubic spline technique. *Sci Inquiry Rev*. 2021;5(3):60–76. https://doi.org/10.32350/sir/53.05