Scientific Inquiry and Review (SIR)
Volume 3, Issue 3, August 2019
ISSN (P): 2521-2427, ISSN (E): 2521-2435
Journal DOI: https://doi.org/10.32350/sir
Issue DOI: https://doi.org/10.32350/sir.33
Homepage: https://ssc.umt.edu.pk/sir/Home.aspx

Journal QR Code:

Article	**Two New Equilateral Triangles Associated with a Triangle**	Indexing Agency
Author(s)	**Dasari Naga Vijay Krishna**	Crossref
Online Published	August 2019	Ulrich's Periodicals Directory
Article DOI	https://doi.org/10.32350/sir.33.04	OCLC WorldCat
Article QR Code	Dasari V N Krishna	Genamics JournalSeek
To cite this article	Krishna DNV. Two new equilateral triangles associated with a triangle. *Sci Inquiry Rev.* 2019;3(3):43–54. Crossref	ROAD Directory of Open Access Scholarly Resources
		Academic Resource Index ResearchBib
Copyright Information	This article is open access and is distributed under the terms of Creative Commons Attribution – Share Alike 4.0 International License.	J-Gate

A publication of the
School of Science, University of Management and Technology
Lahore, Pakistan.

Two New Equilateral Triangles Associated with a Triangle

Dasari Naga Vijay Krishna
Department of Mathematics, Sri Chaitanya Educational Institutions
Machilipatnam, Bengalore, India

vijay9290009015@gmail.com

Abstract

In this short paper, we study two new equilateral triangles associated with an arbitrary triangle and further generalizations.

Keywords: arbitrary triangle, equilateral triangle

Introduction

In this article [1], we discuss two equilateral triangles associated with an arbitrary hexagon. Our aim is to study two new equilateral triangles associated with an arbitrary triangle. In conclusion, we discuss few generalizations in similar configuration.

Before going into the details of the theorems which we prove in this article, let us spend a few minutes discussing the prerequisites of complex numbers used to prove these theorems.

2. Prerequisites of Complex Numbers

2.1 Distance between Two Points

Suppose that complex numbers z_1 and z_2 have the geometric images M_1 and M_2. Then, the distance between the points M_1 and M_2 is given by $M_1M_2 = |z_1 - z_2|$.

2.2 Angle between Two Lines

Recall that a triangle is oriented if an ordering of its vertices is specified. It is positively or directly oriented if the vertices are oriented counter clockwise. Otherwise, the triangle is negatively oriented. Consider two distinct points $M_1(z_1)$ and $M_2(z_2)$ other than the origin of a complex plane. The angle M_1OM_2 is oriented if the points M_1 and M_2 are ordered counter clockwise.

The measure of the directly oriented angle M_1OM_2 equals $\arg \frac{z_1}{z_2}$.

Consider four distinct points: $M_i\ (z_i\ ),\ i \in \{1, 2, 3, 4\}$. The measure of the angle determined by the lines M_1M_3 and M_2M_4 equals $\arg\frac{z_3 - z_1}{z_4 - z_2}$ or $\arg\frac{z_4 - z_2}{z_3 - z_1}$

2.3 Equilateral Triangle on a Segment

Let the points *A* and *B* have affixes *a* and *b*, respectively.

We shall find the affix of the point *C* for which *ABC* is an equilateral triangle with base angle 60^0 and apex C. The midpoint of *AB* has an affix $\frac{(a+b)}{2}$.

The distance from this midpoint to *C* is equal to $\frac{\sqrt{3}\,|AB|}{2}$.

With this we find the affix for *C* as follows,

$$c = \left(\frac{a+b}{2}\right) + i\sqrt{3}\left(\frac{b-a}{2}\right) = \left(\frac{1-i\sqrt{3}}{2}\right)a + \left(\frac{1+i\sqrt{3}}{2}\right)b = \bar{\chi}a + \chi b$$

where $\bar{\chi} = \left(\frac{1-i\sqrt{3}}{2}\right), \chi = \left(\frac{1+i\sqrt{3}}{2}\right)$

Clearly χ is the sixth root of unity $\chi = \left(\frac{1+i\sqrt{3}}{2}\right) = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}$

.

This number is the sixth root of unity since it satisfies $\chi^6 = e^{2i\pi} = \cos 2\pi + \sin 2\pi = 1$. It also satisfies $\chi^3 = -1$ and $\chi.\bar{\chi} = \chi + \bar{\chi} = 1$.

Depending on orientation one can find two vertices *C* that together with *AB* form an equilateral triangle, for which we have respectively $c = \chi a + \bar{\chi} b$ (negative orientation) and $c = \bar{\chi} a + \chi b$ (positive orientation).

From this one easily derives

Lemma 1:

The complex numbers a, b and c are affixes of an equilateral triangle if and only if $a + \chi^2 b + \chi^4 c = 0$ for positive orientation or

$a+\chi^4 b+\chi^2 c=0$ *for negative orientation.*

Lemma 2:

Let z_1, z_2, z_3 be distinct complex numbers. If z_1, z_2, z_3 be collinear in the complex plane then there exists k such that $z_2=\dfrac{z_3+kz_1}{1+k}$.

In this article [1], the following theorem has been proved.

2.4 Two Equilateral Triangles Associated with a Hexagon

Consider a hexagon $A_1A_2A_3A_4A_5A_6$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the six sides externally, where B_j are the apex of the equilateral triangle constructed on the side A_jA_{j+1}, externally. Here, we take the subscripts modulo 6. Let C_j be the midpoint of B_jB_{j+1}. Let P_1, P_2, P_3 be the points of intersection of the line segments C_1C_4, C_2C_5 *and* C_3C_6. We proceed with the following interesting result.

a) The line segments C_1C_4, C_2C_5 and C_3C_6 are of equal length.

b) $P_1P_2P_3$ forms an equilateral triangle (see Figure -1).

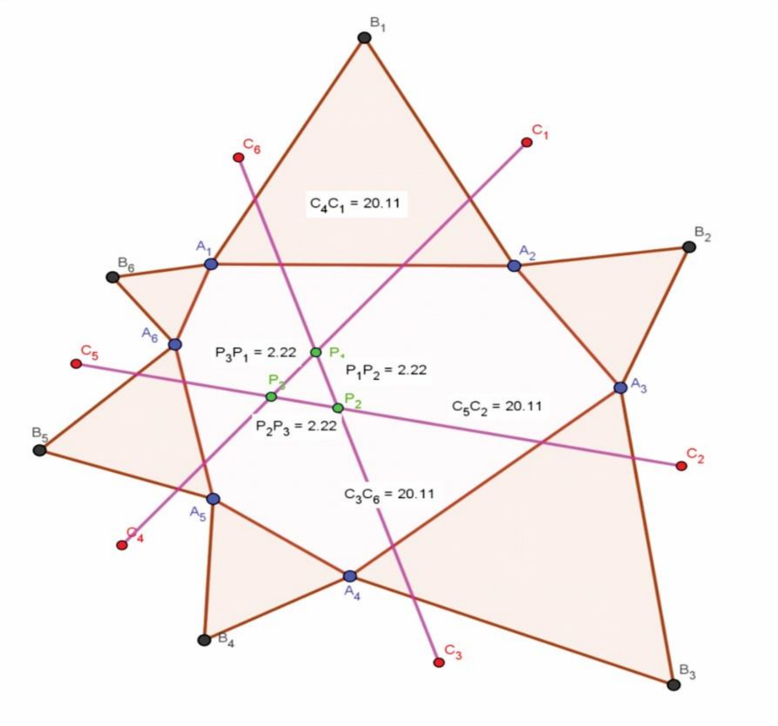

Figure 1.

The above result is true even if equilateral triangles are constructed on the sides of the arbitrary hexagon internally ***(see Figure-2).***

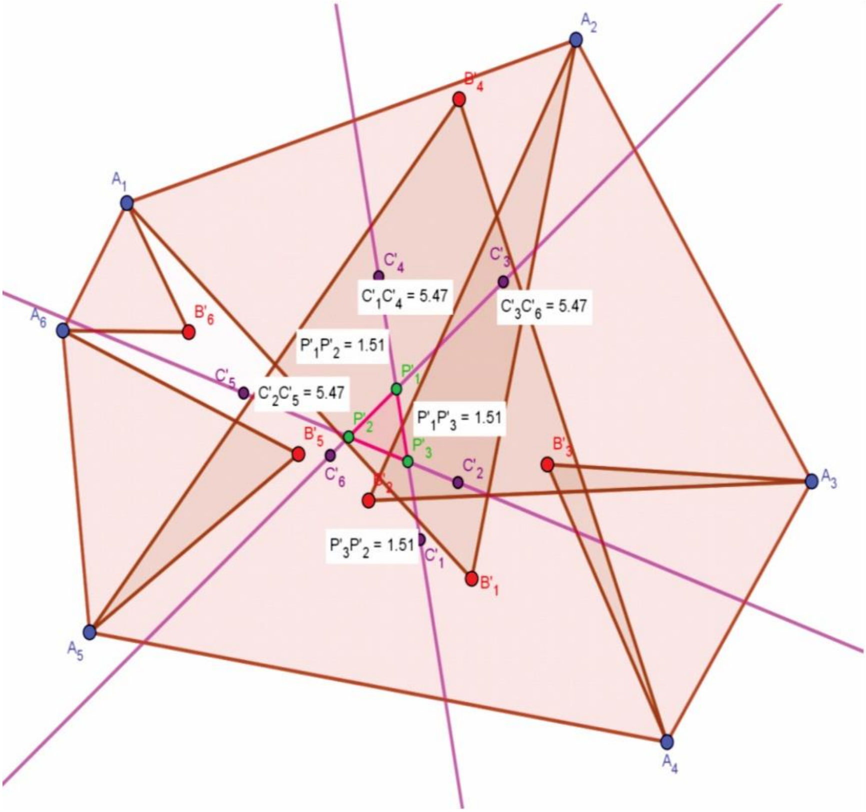

Figure 2.

Let us discuss our main theorem.

2.5 Theorem 1

Two New Equilateral Triangles Associated with a Triangle

Consider an arbitrary triangle $A_1A_3A_5$. With equilateral triangles $B_jA_jA_{j+1}$ constructed on the sides externally, where B_j are the apex of equilateral triangle constructed on the line segment A_jA_{j+1}. Let three arbitrary points A_2, A_4 and A_6 lay on the sides A_1A_3, A_3A_5 and A_5A_1, respectively. Here, we take the subscripts modulo 6. Let C_j *be* the midpoint of B_jB_{j+1}. Let P_1, P_2, P_3 be the points of intersection of the line segments C_1C_4, C_2C_5 and C_3C_6. We proceed with the following interesting result.

a) The line segments C_1C_4, C_2C_5 and C_3C_6 are of equal length.
b) $P_1P_2P_3$ **forms** ***an equilateral triangle (see Figure-3).***

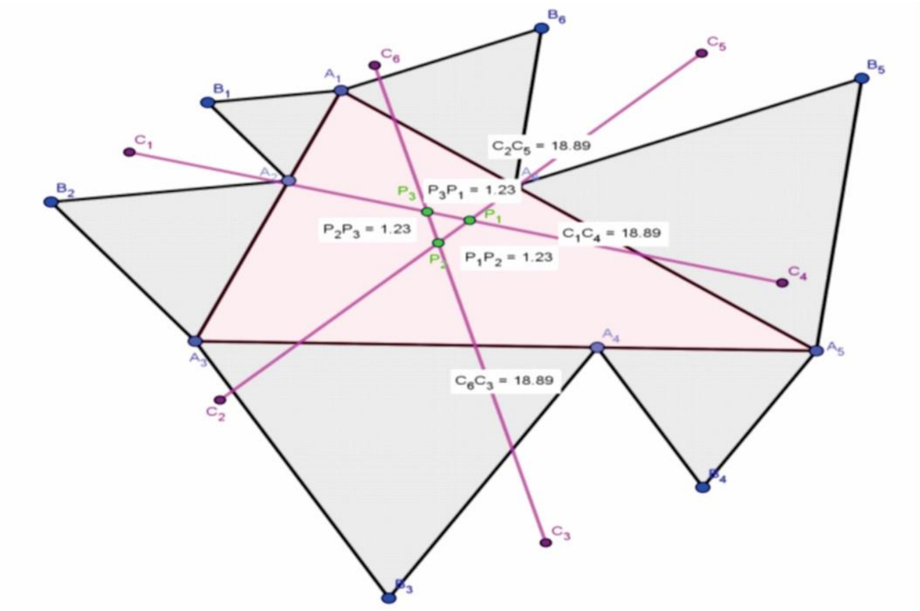

Figure 3.

Proof

Suppose the triangle $A_1A_3A_5$ with the points A_2, A_4 and A_6 is in the complex plane.

Each of the point A_j, j = 1, 2,..,6, has a complex affix α_j and it is clear that α_j, α_{j+1} and α_{j+2} are collinear for j = 1, 3, 5 under modulo 6.

Hence, there exists three real numbers k_1, k_2 and k_3 such that

$$\alpha_2 = \frac{\alpha_3 + k_1\alpha_1}{1+k_1},\ \alpha_4 = \frac{\alpha_5 + k_2\alpha_3}{1+k_2}\ \text{and}\ \alpha_6 = \frac{\alpha_1 + k_3\alpha_5}{1+k_3}\ \text{(using lemma-2)}$$

It is clear using lemma-1, B_j has a complex affix $-\chi^2\alpha_j - \chi^4\alpha_{j+1}$

and Cj has complex affix as $z_j = \dfrac{-\chi^2\alpha_j + \alpha_{j+1} - \chi^4\alpha_{j+2}}{2}$

That is $C_1\,(z_1) = \dfrac{-\chi^2\alpha_1 + \alpha_2 - \chi^4\alpha_3}{2}$,

$C_2(z_2) = \dfrac{-\chi^2\alpha_2 + \alpha_3 - \chi^4\alpha_4}{2}$,

$C_3(z_3) = \dfrac{-\chi^2\alpha_3 + \alpha_4 - \chi^4\alpha_5}{2}$,

$C_4(z_4) = \dfrac{-\chi^2\alpha_4 + \alpha_5 - \chi^4\alpha_6}{2}$,

$C_5(z_5) = \dfrac{-\chi^2\alpha_5 + \alpha_6 - \chi^4\alpha_1}{2}$

and $C_6(z_6) = \dfrac{-\chi^2\alpha_6 + \alpha_1 - \chi^4\alpha_2}{2}$

Now, it is easy to verify that

$C_1C_4 = |z_1 - z_4| = \dfrac{1}{2}|\chi^2(\alpha_4 - \alpha_1) + (\alpha_2 - \alpha_5) + \chi^4(\alpha_6 - \alpha_3)|$

$C_2C_5 = |z_2 - z_5| = |-\chi^2(C_1C_4)| = C_1C_4$ and $|C_3C_6| = |z_3 - z_6| = |\chi^4(C_1C_4)| = C_1C_4$

Hence $C_1C_4 = C_2C_5 = C_3C_6$ which completes the proof of (a).

For (b) we proceed as follows.

If the measure of the angle determined by the lines C_1C_4 and C_2C_5 at P_1 is θ_1 then $\theta_1 = \arg\dfrac{z_4 - z_1}{z_5 - z_2}$ or $\theta_1 = \arg\dfrac{z_5 - z_2}{z_4 - z_1}$

It gives

$$\theta_1 = \arg\frac{z_4 - z_1}{z_5 - z_2} = \arg\left(\frac{\chi^2(\alpha_1 - \alpha_4) + (\alpha_5 - \alpha_2) + \chi^4(\alpha_3 - \alpha_6)}{\chi^2(\alpha_2 - \alpha_5) + (\alpha_6 - \alpha_3) + \chi^4(\alpha_4 - \alpha_1)}\right) = \arg\left(\frac{-1}{\chi^2}\right) = \arg(-\chi^4) = 60^0$$

In a similar way, if the measure of the angle determined by the lines C_2C_5 and $C3C6$ at $P2$ is $\theta 2$, while $C3C6$ and $C1C4$ at $P3$ is $\theta 3$, we can prove that $\theta_2 = \theta_3 = 60^0$

Hence, triangle $P_1P_2P_3$ is an equilateral triangle, which proves (b).

By replacing χ^2 by χ^4 and χ^4 by χ^2 in the proof of Theorem 1, we have an analogous result of Theorem 1 with an equilateral triangle constructed on the sides of the given triangle, internally.

In other words, consider an arbitrary triangle $A_1A_3A_5$. With equilateral triangles $B'_jA_jA_{j+1}$ constructed on the sides externally, where B'_j are the apex of the equilateral triangle constructed on the line segment A_jA_{j+1}. Let A_2, A_4 and A_6 be three arbitrary points lying on the sides A_1A_3, A_3A_5 and A_5A_1, respectively. Here, we take the subscripts modulo 6. Let

C'_j be the midpoint of $B'_j B'_{j+1}$. Let P'_1, P'_2 and P'_3 be the points of intersection of the line segments $C'_1C'_4$, $C'_2C'_5$ and $C'_3C'_6$. Then,

a) The line segments $C'_1C'_4$***,*** $C'_2C'_5$ ***and*** $C'_3C'_6$ ***are of equal length.***

b) P'_1 P'_2 P'_3 ***forms an equilateral triangle*** (see Figure 4).

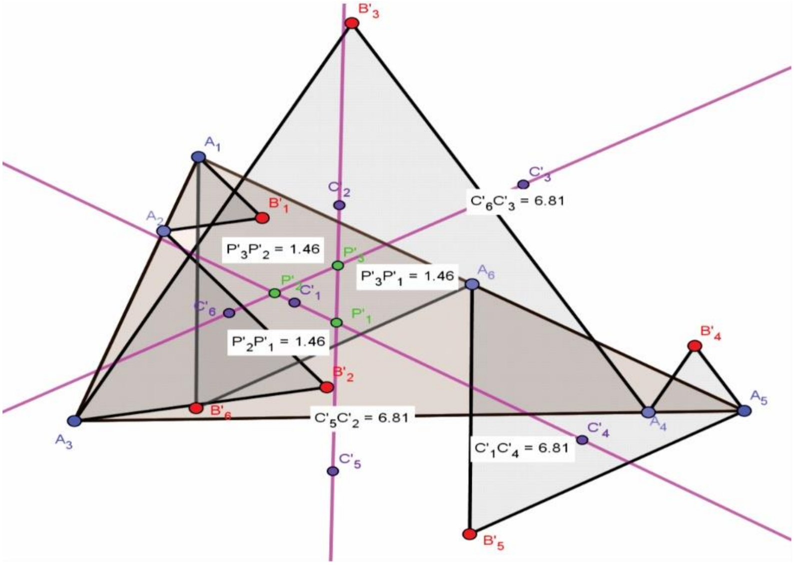

Figure 4.

Note 1

If we observe closely Theorem 1 and the theorem we proved in [1], there is not much difference in proving both of them. This forces us to generalize the statement that if some interesting property is true for a polygon of 6n vertices then the same property is valid for an arbitrary triangle by considering the remaining 6n-3 vertices of the polygon on the sides of the triangle, such that 2n-1 vertices as a point on each side of the triangle.

Below we list out some generalizations about the same configuration which can be demonstrated using the same ideas from the statements in this article.

3. Generalizations

3.1 Proposition 1

Given an octagon $A_1A_2A_3A_4A_5A_6\ A_7A_8$ *with equilateral triangles* $B_jA_jA_{j+1}$ *constructed on the sides externally or internally, where* B_j *are the apex* of equilateral triangles constructed on the side A_jA_{j+1}, externally or internally. Here, we take the subscripts modulo 8. Let C_j be the midpoints of B_jB_{j+1}. If P_j are the midpoints of the line segment C_jC_{j+4}, then the quadrilateral $P_1P_2P_3P_4$ is a parallelogram and the point of intersection of diagonals of both parallelograms (external and internal cases) coincide with each other (see Figure 5).

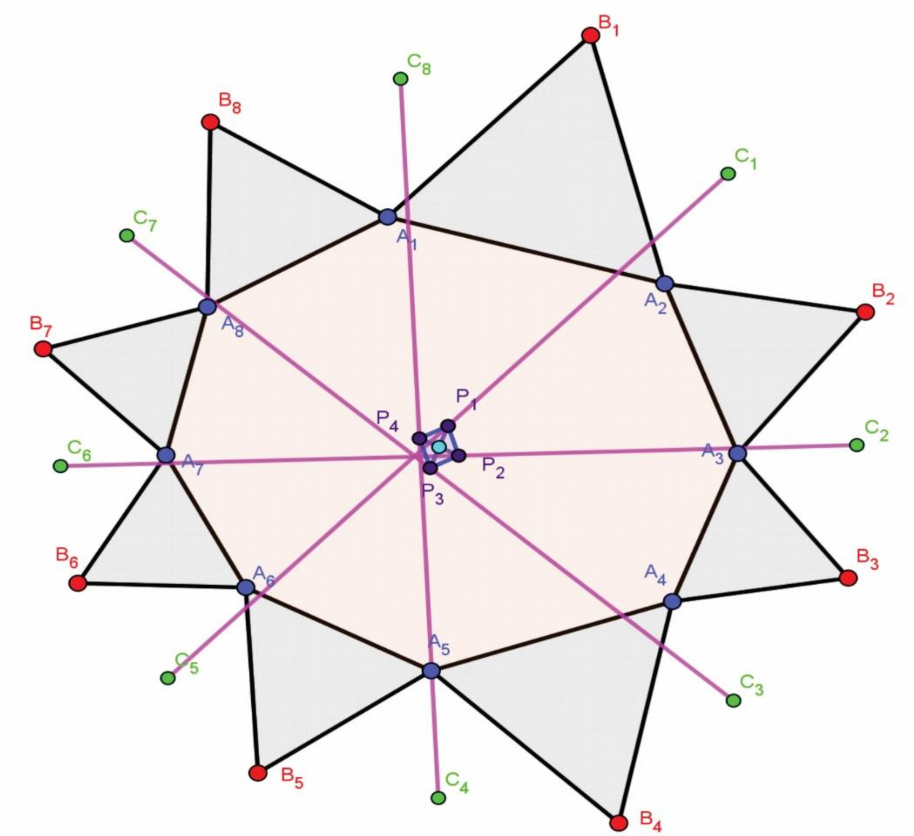

Figure 5.

3.2 Proposition 2

Given an octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ construct squares $B_1A_1A_2B_2$, $B_3A_2A_3B_4$, $B_5A_3A_4B_6$, $B_7A_4A_5B_8$, $B_9A_5A_6B_{10}$, $B_{11}A_6A_7B_{12}$, $B_{13}A_7A_8B_{14}$, and B15A8A1B16 on the sides of hexagon externally or internally. Let $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ be the midpoints of B_2B_3, B_4B_5, B_6B_7, B_8B_9, $B_{10}B_{11}$, $B_{12}B_{13}$, $B_{14}B_{15}$, $B_{16}B_1$,. If Pj are the midpoints of the line segment C_jC_{j+4}, then the quadrilateral $P_1P_2P_3P_4$ is an Iso Ortho diagonal quadrilateral (see Figure 6).

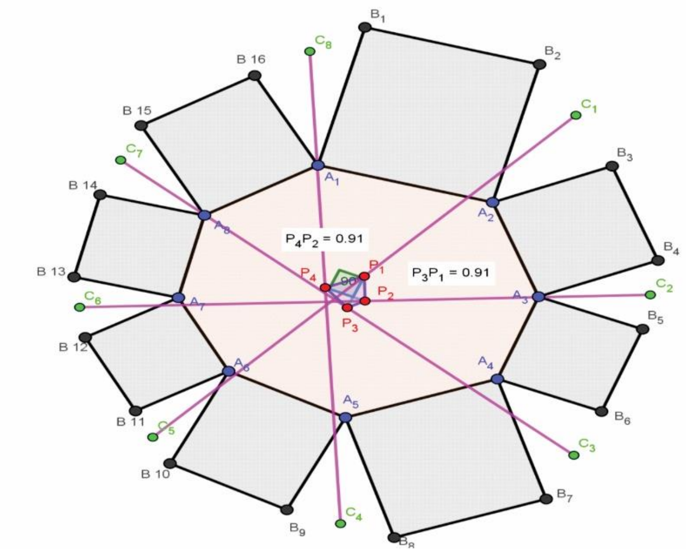

Figure 6.

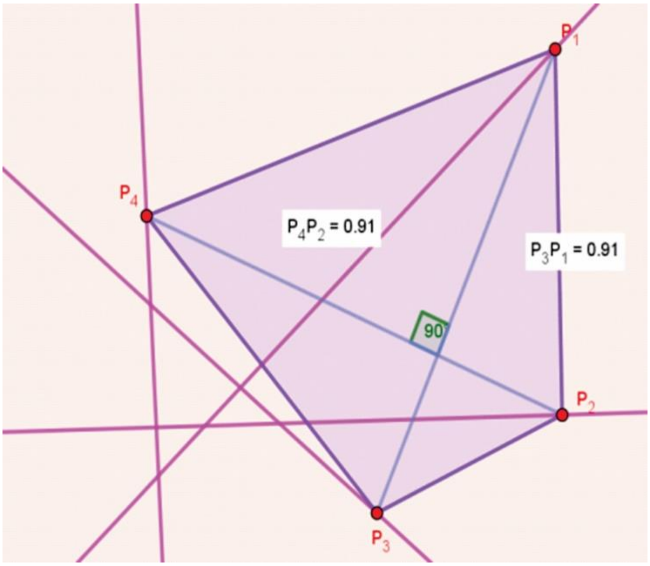

Figure 6[a].

3.3 Proposition 3

Given a 12-gon A1A2A3A4A5A6A7A8 $A_9A_{10}A_{11}A_{12}$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the sides externally or internally, where B_j are the apex of equilateral triangles constructed on the side A_jA_{j+1}, externally or internally. Here, we take the subscripts modulo 12. Let C_j be the midpoints of B_jB_{j+1}. Let P_j be the midpoints of the line segment

C_jC_{j+4}. If Q_1, Q_2, Q_3 are the points of intersection of the line segments P_1P_4, P_2P_5 and P_3P_6, then

(a) The line segments P_1P_4, P_2P_5 and P_3P_6 are equal in length.
(b)Triangle $Q_1Q_2Q_3$ forms an equilateral triangle (see Figure -7, 7A).

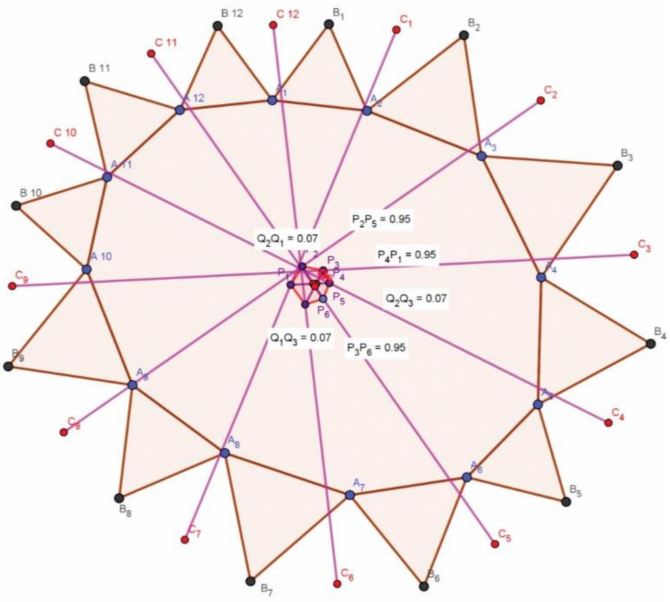

Figure 7.

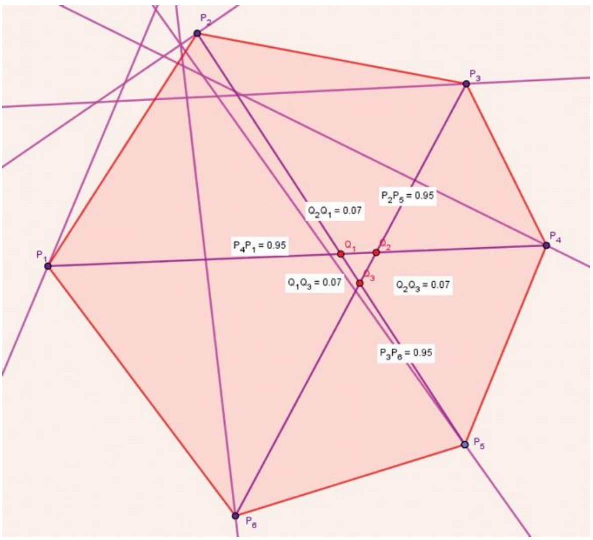

Figure 7[a].

3.4 Proposition 4

According to Note 1 and Proposition 3, we can state that Proposition 3 is also true when 12-gon (6(2) vertices) is transformed to a triangle by considering the remaining 12-3=9 vertices as three arbitrary points on each side of the triangle.

For further study regarding these types of equilateral triangles, we can refer to [2, 3, 4].

Acknowledgement

The author would like to thank an anonymous referee for his/her kind comments and suggestions, which lead to a better presentation of this paper.

References

[1] Krishna DNV. An equilateral triangle associated with a hexagon. *Math Gaz*. 2018;102(555): 515–516. doi.org/10.1017/mag.2018.128

[2] Oai DT. Equilateral triangles and Kiepert perspectors in complex numbers. *Forum Geometricorum*. 2015;15: 105–114.

[3] Oai DT. Some new equilateral triangles in a plane geometry. *Global J Adv Res Classical Mod Geometries*. 2018;7(2): 73–91.

[4] Oai DT. Some equilateral triangles perspective to the reference triangle ABC. *Int J Comput Discovered Math (IJCDM)*. 2018;3: 88–96.