
Scientific Inquiry and Review (SIR)
Volume 3, Issue 3, August 2019
ISSN (P): 2521-2427, ISSN (E): 2521-2435
Journal DOI: https://doi.org/10.32350/sir
Issue DOI: https://doi.org/10.32350/sir.33
Homepage: https://ssc.umt.edu.pk/sir/Home.aspx

Journal QR Code:


| | |
|---|---|
| Article | **Two New Equilateral Triangles Associated with a Triangle** |
| Author(s) | **Dasari Naga Vijay Krishna** |
| Online Published | August 2019 |
| Article DOI | https://doi.org/10.32350/sir.33.04 |
| Article QR Code | Dasari V N Krishna |
| To cite this article | Krishna DNV. Two new equilateral triangles associated with a triangle. *Sci Inquiry Rev.* 2019;3(3):43–54. Crossref |
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A publication of the
School of Science, University of Management and Technology
Lahore, Pakistan.

# Two New Equilateral Triangles Associated with a Triangle

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## Abstract

*In this short paper, we study two new equilateral triangles associated with an arbitrary triangle and further generalizations.*

***Keywords:*** arbitrary triangle, equilateral triangle

## Introduction

In this article [1], we discuss two equilateral triangles associated with an arbitrary hexagon. Our aim is to study two new equilateral triangles associated with an arbitrary triangle. In conclusion, we discuss few generalizations in similar configuration.

Before going into the details of the theorems which we prove in this article, let us spend a few minutes discussing the prerequisites of complex numbers used to prove these theorems.

## 2. Prerequisites of Complex Numbers

### 2.1 Distance between Two Points

Suppose that complex numbers $z_1$ and $z_2$ have the geometric images $M_1$ and $M_2$. Then, the distance between the points $M_1$ and $M_2$ is given by $M_1M_2 = |z_1 - z_2|$.

### 2.2 Angle between Two Lines

Recall that a triangle is oriented if an ordering of its vertices is specified. It is positively or directly oriented if the vertices are oriented counter clockwise. Otherwise, the triangle is negatively oriented. Consider two distinct points $M_1(z_1)$ and $M_2(z_2)$ other than the origin of a complex plane. The angle $M_1OM_2$ is oriented if the points $M_1$ and $M_2$ are ordered counter clockwise.

The measure of the directly oriented angle $M_1OM_2$ equals $\arg \frac{z_1}{z_2}$.

Consider four distinct points: $M_i\ (z_i\ ),\ i \in \{1, 2, 3, 4\}$. The measure of the angle determined by the lines $M_1M_3$ and $M_2M_4$ equals $\arg\frac{z_3 - z_1}{z_4 - z_2}$ or $\arg\frac{z_4 - z_2}{z_3 - z_1}$

### 2.3 Equilateral Triangle on a Segment

Let the points *A* and *B* have affixes *a* and *b*, respectively.

We shall find the affix of the point *C* for which *ABC* is an equilateral triangle with base angle $60^0$ and apex C. The midpoint of *AB* has an affix $\frac{(a+b)}{2}$.

The distance from this midpoint to *C* is equal to $\frac{\sqrt{3}\,|AB|}{2}$.

With this we find the affix for *C* as follows,

$$c = \left(\frac{a+b}{2}\right) + i\sqrt{3}\left(\frac{b-a}{2}\right) = \left(\frac{1-i\sqrt{3}}{2}\right)a + \left(\frac{1+i\sqrt{3}}{2}\right)b = \bar{\chi}a + \chi b$$

where $\bar{\chi} = \left(\frac{1-i\sqrt{3}}{2}\right), \chi = \left(\frac{1+i\sqrt{3}}{2}\right)$

Clearly $\chi$ is the sixth root of unity $\chi = \left(\frac{1+i\sqrt{3}}{2}\right) = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}$

.

This number is the sixth root of unity since it satisfies $\chi^6 = e^{2i\pi} = \cos 2\pi + \sin 2\pi = 1$. It also satisfies $\chi^3 = -1$ and $\chi \cdot \bar{\chi} = \chi + \bar{\chi} = 1$.

Depending on orientation one can find two vertices *C* that together with *AB* form an equilateral triangle, for which we have respectively $c = \chi a + \bar{\chi} b$ (negative orientation) and $c = \bar{\chi} a + \chi b$ (positive orientation).

From this one easily derives

**Lemma 1:**

The complex numbers a, b and c are affixes of an equilateral triangle if and only if $a + \chi^2 b + \chi^4 c = 0$ for positive orientation or

$a+\chi^4 b+\chi^2 c=0$ *for negative orientation.*

**Lemma 2:**

Let $z_1$, $z_2$, $z_3$ be distinct complex numbers. If $z_1$, $z_2$, $z_3$ be collinear in the complex plane then there exists k such that $z_2=\frac{z_3+kz_1}{1+k}$.

In this article [1], the following theorem has been proved.

## 2.4 Two Equilateral Triangles Associated with a Hexagon

Consider a hexagon $A_1A_2A_3A_4A_5A_6$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the six sides externally, where $B_j$ are the apex of the equilateral triangle constructed on the side $A_jA_{j+1}$, externally. Here, we take the subscripts modulo 6. Let $C_j$ be the midpoint of $B_jB_{j+1}$. Let $P_1$, $P_2$, $P_3$ be the points of intersection of the line segments $C_1C_4$, $C_2C_5$ *and* $C_3C_6$. We proceed with the following interesting result.

***a) The line segments $C_1C_4$, $C_2C_5$ and $C_3C_6$ are of equal length.***

***b) $P_1P_2P_3$ forms an equilateral triangle (see Figure -1).***

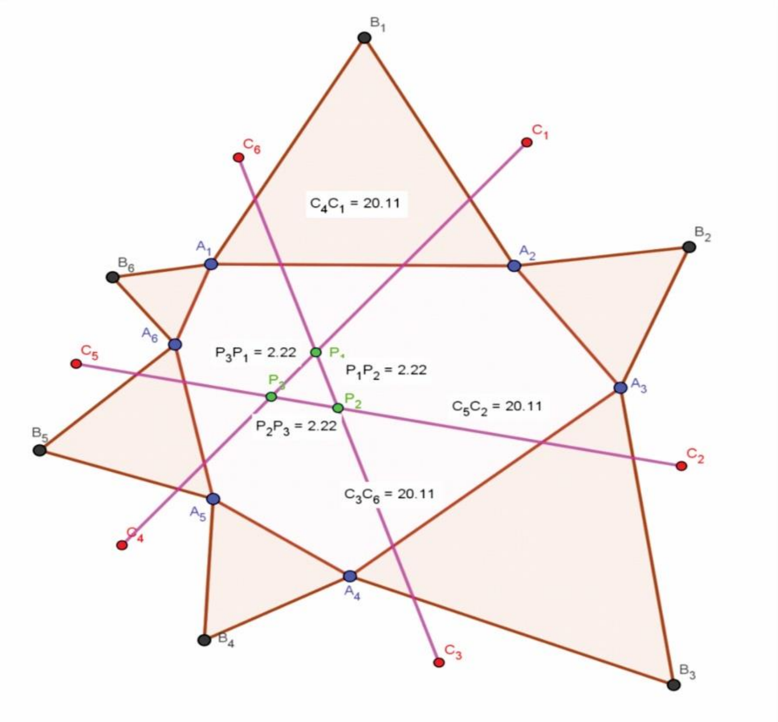


**Figure 1.**

The above result is true even if equilateral triangles are constructed on the sides of the arbitrary hexagon internally ***(see Figure-2).***

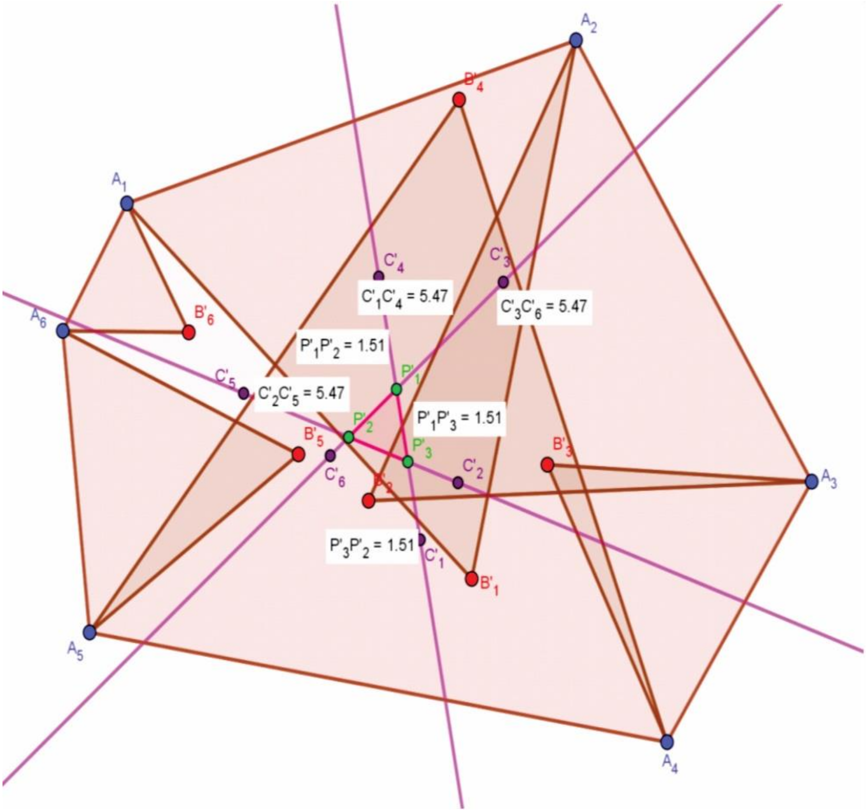


**Figure 2.**

Let us discuss our main theorem.

**2.5 Theorem 1**

**Two New Equilateral Triangles Associated with a Triangle**

Consider an arbitrary triangle $A_1A_3A_5$. With equilateral triangles $B_jA_jA_{j+1}$ constructed on the sides externally, where $B_j$ are the apex of equilateral triangle constructed on the line segment $A_jA_{j+1}$. Let three arbitrary points $A_2$, $A_4$ and $A_6$ lay on the sides $A_1A_3$, $A_3A_5$ and $A_5A_1$, respectively. Here, we take the subscripts modulo 6. Let $C_j$ *be* the midpoint of $B_jB_{j+1}$. Let $P_1$, $P_2$, $P_3$ be the points of intersection of the line segments $C_1C_4$, $C_2C_5$ and $C_3C_6$. We proceed with the following interesting result.

***a) The line segments $C_1C_4$, $C_2C_5$ and $C_3C_6$ are of equal length.***
***b) $P_1P_2P_3$* forms *an equilateral triangle (see Figure-3).***

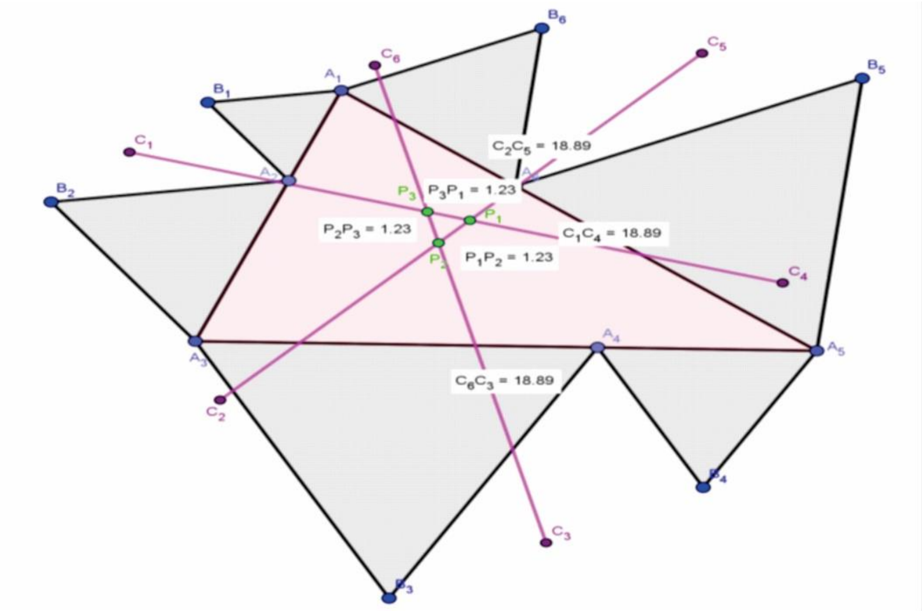


**Figure 3.**

**Proof**

Suppose the triangle $A_1A_3A_5$ with the points $A_2$, $A_4$ and $A_6$ is in the complex plane.

Each of the point $A_j$, $j = 1, 2,..,6$, has a complex affix $\alpha_j$ and it is clear that $\alpha_j$, $\alpha_{j+1}$ and $\alpha_{j+2}$ are collinear for $j = 1, 3, 5$ under modulo 6.

Hence, there exists three real numbers $k_1$, $k_2$ and $k_3$ such that

$$\alpha_2 = \frac{\alpha_3 + k_1\alpha_1}{1+k_1},\ \alpha_4 = \frac{\alpha_5 + k_2\alpha_3}{1+k_2}\ \textit{and}\ \alpha_6 = \frac{\alpha_1 + k_3\alpha_5}{1+k_3}\ \text{(using lemma-2)}$$

It is clear using lemma-1, $B_j$ has a complex affix $-\chi^2\alpha_j - \chi^4\alpha_{j+1}$

and Cj has complex affix as $z_j = \dfrac{-\chi^2\alpha_j + \alpha_{j+1} - \chi^4\alpha_{j+2}}{2}$

That is $C_1\,(z_1) = \dfrac{-\chi^2\alpha_1 + \alpha_2 - \chi^4\alpha_3}{2}$,

$C_2(z_2) = \dfrac{-\chi^2\alpha_2 + \alpha_3 - \chi^4\alpha_4}{2}$,

$C_3(z_3) = \dfrac{-\chi^2\alpha_3 + \alpha_4 - \chi^4\alpha_5}{2}$,

$C_4(z_4) = \dfrac{-\chi^2\alpha_4+\alpha_5-\chi^4\alpha_6}{2}$,

$C_5(z_5) = \dfrac{-\chi^2\alpha_5+\alpha_6-\chi^4\alpha_1}{2}$

and $C_6(z_6) = \dfrac{-\chi^2\alpha_6+\alpha_1-\chi^4\alpha_2}{2}$

Now, it is easy to verify that

$C_1C_4 = |z_1 - z_4| = \frac{1}{2}|\chi^2(\alpha_4-\alpha_1)+(\alpha_2-\alpha_5)+\chi^4(\alpha_6-\alpha_3)|$

$C_2C_5 = |z_2 - z_5| = |-\chi^2(C_1C_4)| = C_1C_4$ and $|C_3C_6| = |z_3 - z_6| = |\chi^4(C_1C_4)| = C_1C_4$

Hence $C_1C_4 = C_2C_5 = C_3C_6$ which completes the proof of (a).

For (b) we proceed as follows.

If the measure of the angle determined by the lines $C_1C_4$ and $C_2C_5$ at $P_1$ is $\theta_1$ then $\theta_1 = \arg\dfrac{z_4-z_1}{z_5-z_2}$ or $\theta_1 = \arg\dfrac{z_5-z_2}{z_4-z_1}$

It gives

$$\theta_1 = \arg\frac{z_4-z_1}{z_5-z_2} = \arg\left(\frac{\chi^2(\alpha_1-\alpha_4)+(\alpha_5-\alpha_2)+\chi^4(\alpha_3-\alpha_6)}{\chi^2(\alpha_2-\alpha_5)+(\alpha_6-\alpha_3)+\chi^4(\alpha_4-\alpha_1)}\right) = \arg\left(\frac{-1}{\chi^2}\right) = \arg(-\chi^4) = 60^0$$

In a similar way, if the measure of the angle determined by the lines $C_2C_5$ and $C3C6$ at $P2$ is $\theta2$ , while $C3C6$ and $C1C4$ at $P3$ is $\theta3$, we can prove that $\theta_2 = \theta_3 = 60^0$

Hence, triangle $P_1P_2P_3$ is an equilateral triangle, which proves (b).

By replacing $\chi^2$ by $\chi^4$ and $\chi^4$ by $\chi^2$ in the proof of Theorem 1, we have an analogous result of Theorem 1 with an equilateral triangle constructed on the sides of the given triangle, internally.

In other words, consider an arbitrary triangle $A_1A_3A_5$. With equilateral triangles $B'_jA_jA_{j+1}$ constructed on the sides externally, where $B'_j$ are the apex of the equilateral triangle constructed on the line segment $A_jA_{j+1}$. Let $A_2$, $A_4$ and $A_6$ be three arbitrary points lying on the sides $A_1A_3$, $A_3A_5$ and $A_5A_1$, respectively. Here, we take the subscripts modulo 6. Let

$C'_j$ be the midpoint of $B'_j B'_{j+1}$. Let $P'_1$, $P'_2$ and $P'_3$ be the points of intersection of the line segments $C'_1C'_4$, $C'_2C'_5$ and $C'_3C'_6$. Then,

***a) The line segments $C'_1C'_4$, $C'_2C'_5$ and $C'_3C'_6$ are of equal length.***

***b) $P'_1\ P'_2\ P'_3$ forms an equilateral triangle*** (see Figure 4).

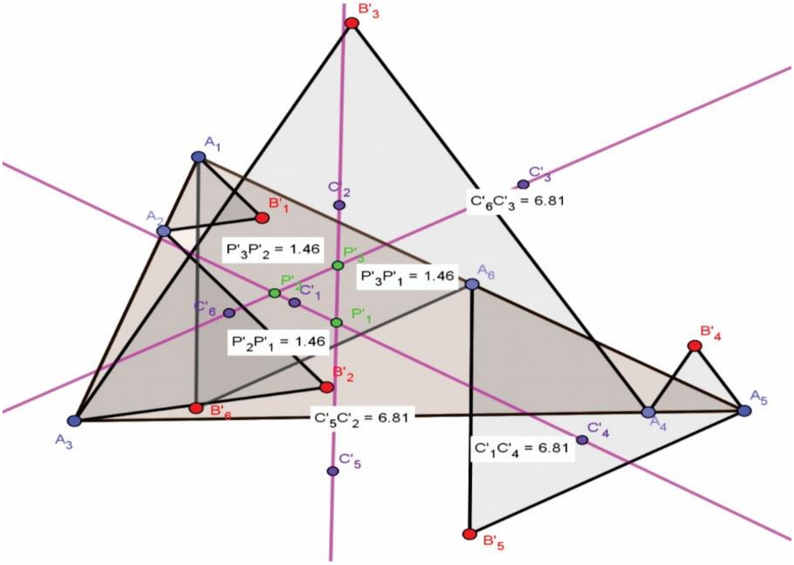


**Figure 4.**

**Note 1**

If we observe closely Theorem 1 and the theorem we proved in [1], there is not much difference in proving both of them. This forces us to generalize the statement that if some interesting property is true for a polygon of 6n vertices then the same property is valid for an arbitrary triangle by considering the remaining 6n-3 vertices of the polygon on the sides of the triangle, such that 2n-1 vertices as a point on each side of the triangle.

Below we list out some generalizations about the same configuration which can be demonstrated using the same ideas from the statements in this article.

## 3. Generalizations

### 3.1 Proposition 1

*Given an octagon $A_1A_2A_3A_4A_5A_6\ A_7A_8$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the sides externally or internally, where $B_j$ are the apex* of equilateral triangles constructed on the side $A_jA_{j+1,}$ externally or internally. Here, we take the subscripts modulo 8. Let $C_j$ be the midpoints of $B_jB_{j+1}$. If $P_j$ are the midpoints of the line segment $C_jC_{j+4,}$ then the quadrilateral $P_1P_2P_3P_4$ is a parallelogram and the point of intersection of diagonals of both parallelograms (external and internal cases) coincide with each other (see Figure 5).

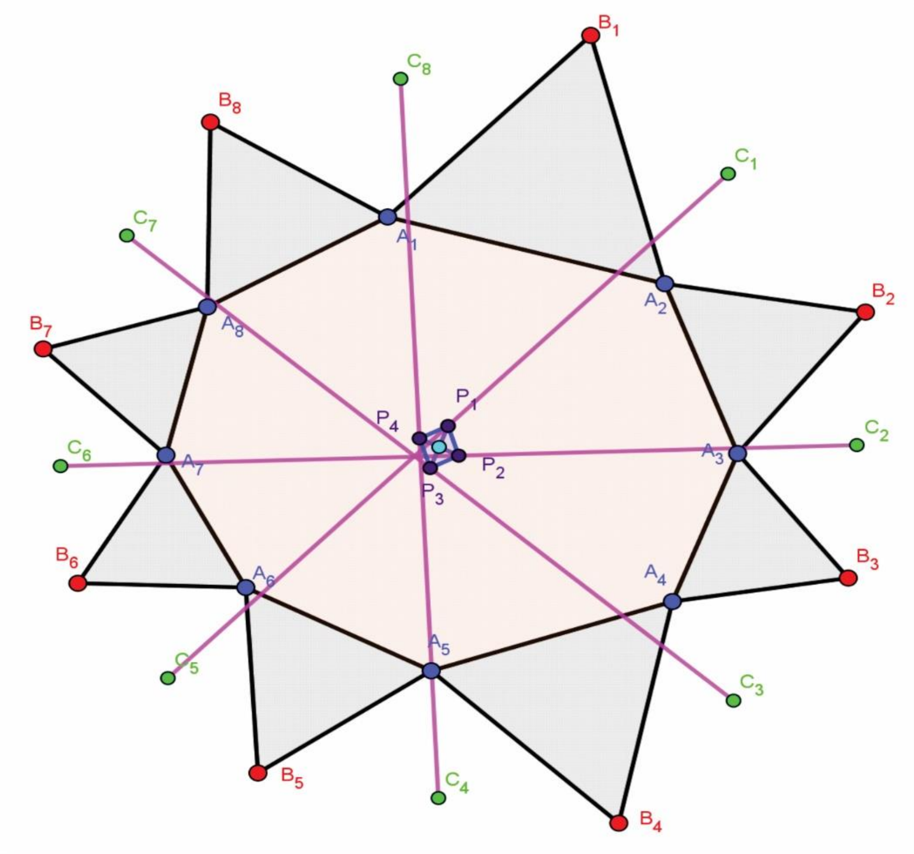


**Figure 5.**

## 3.2 Proposition 2

Given an octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ construct squares $B_1A_1A_2B_2$, $B_3A_2A_3B_4$, $B_5A_3A_4B_6$, $B_7A_4A_5B_8$, $B_9A_5A_6B_{10}$, $B_{11}A_6A_7B_{12}$, $B_{13}A_7A_8B_{14}$, and B15A8A1B16 on the sides of hexagon externally or internally. Let $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$, $C_7$, $C_8$ be the midpoints of $B_2B_3$, $B_4B_5$, $B_6B_7$, $B_8B_9$, $B_{10}B_{11}$, $B_{12}B_{13}$, $B_{14}B_{15}$, $B_{16}B_1$,. If Pj are the midpoints of the line segment $C_jC_{j+4,}$ then the quadrilateral $P_1P_2P_3P_4$ is an Iso Ortho diagonal quadrilateral (see Figure 6).

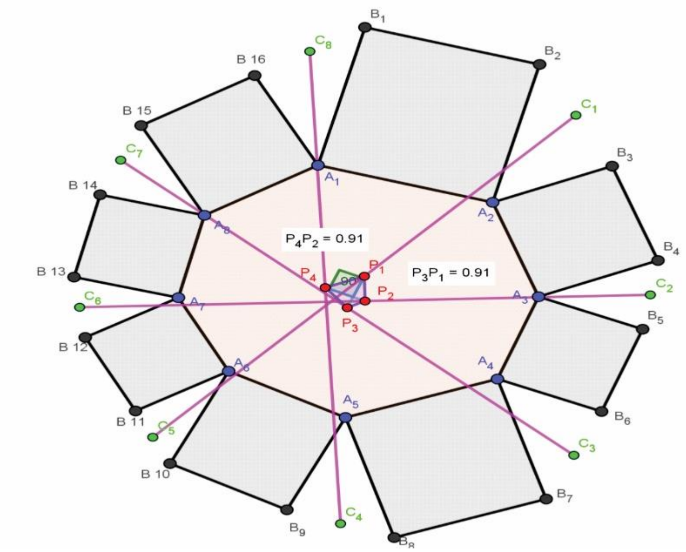


**Figure 6.**

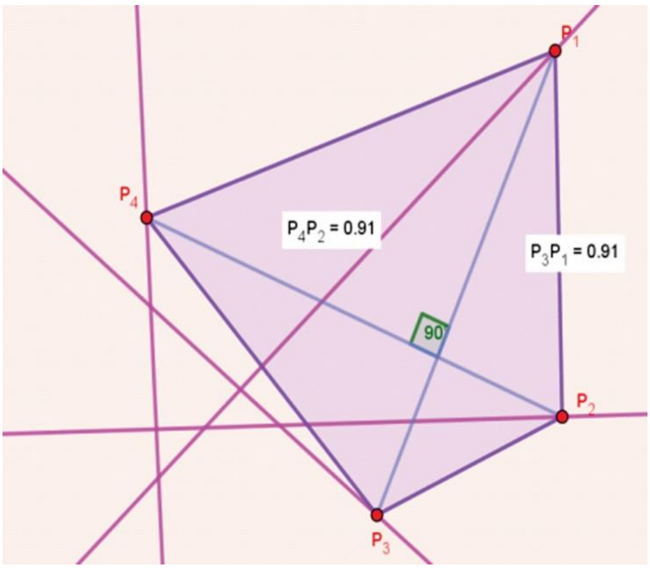


**Figure 6$^a$.**

### 3.3 Proposition 3

Given a 12-gon A1A2A3A4A5A6A7A8 $A_9A_{10}A_{11}A_{12}$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the sides externally or internally, where $B_j$ are the apex of equilateral triangles constructed on the side $A_jA_{j+1,}$ externally or internally. Here, we take the subscripts modulo 12. Let $C_j$ be the midpoints of $B_jB_{j+1}$. Let $P_j$ be the midpoints of the line segment

*$C_jC_{j+4}$. If $Q_1$, $Q_2$, $Q_3$ are the points of intersection of the line segments $P_1P_4$, $P_2P_5$ and $P_3P_6$, then*

(a) The line segments $P_1P_4$, $P_2P_5$ and $P_3P_6$ are equal in length.
(b)Triangle $Q_1Q_2Q_3$ forms an equilateral triangle (see Figure -7, 7A).

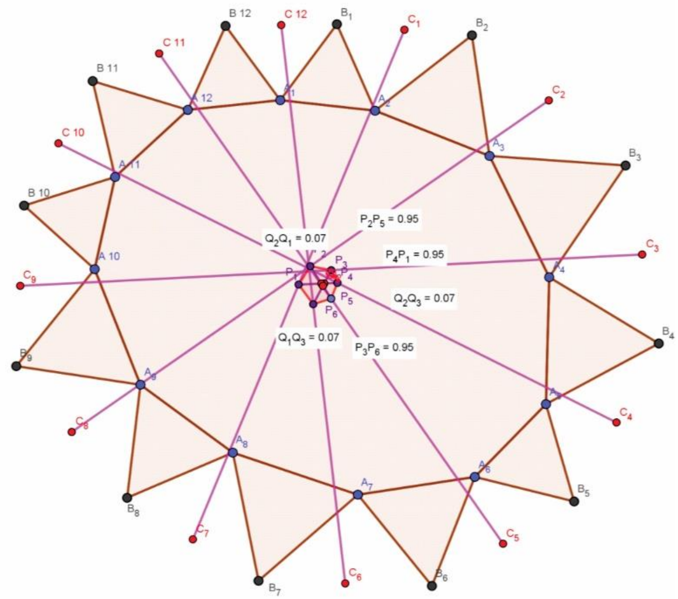


**Figure 7.**

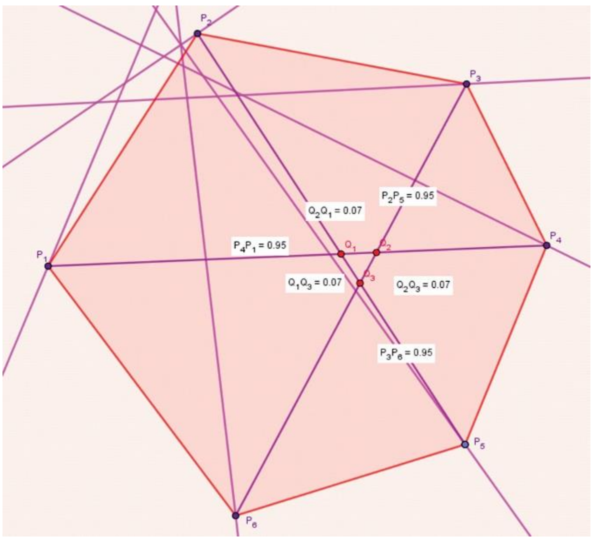


**Figure 7$^a$.**

### 3.4 Proposition 4

According to Note 1 and Proposition 3, we can state that Proposition 3 is also true when 12-gon (6(2) vertices) is transformed to a triangle by considering the remaining 12-3=9 vertices as three arbitrary points on each side of the triangle.

For further study regarding these types of equilateral triangles, we can refer to [2, 3, 4].

**Acknowledgement**

The author would like to thank an anonymous referee for his/her kind comments and suggestions, which lead to a better presentation of this paper.

## References

[1] Krishna DNV. An equilateral triangle associated with a hexagon. *Math Gaz*. 2018;102(555): 515–516. doi.org/10.1017/mag.2018.128

[2] Oai DT. Equilateral triangles and Kiepert perspectors in complex numbers. *Forum Geometricorum*. 2015;15: 105–114.

[3] Oai DT. Some new equilateral triangles in a plane geometry. *Global J Adv Res Classical Mod Geometries*. 2018;7(2): 73–91.

[4] Oai DT. Some equilateral triangles perspective to the reference triangle ABC. *Int J Comput Discovered Math (IJCDM)*. 2018;3: 88–96.