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Coefficient Inequalities for Certain Subclass of Starlike Function with respect to Symmetric points related to *q*-exponential Function

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ABSTRACT

The current study aims to define a class of starlike functions with respect to symmetric points subordinated with q-exponential functions. Furthermore, to investigate the coefficient inequalities and possible upper-bound of the third-order Hankel determinant for the functions belonging to our new class this study observed the new and already derived results for further analysis.

Keywords: analytic function, Hankel Determinant, q-derivative, symmetric point

1. INTRODUCTION

The analytic functions, also called holomorphic functions, are complex-valued functions that are defined and differentiable at every point within their domain of definition. The class of all analytical functions f with the normalized condition in the open unit disc $E = \{z{:}|z| < 1\}$ is asymbol by A and has the Taylor series, which is stated as:

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{1.1}$$

The class of univalent and analytical function unit disc E is proved by S. Caratheodory functions are a class of complex-valued functions, which are defined on a domain in the complex plane. They are named after the mathematician Constant in Caratheodory symbolized by and the function of this class is of the form

$$p(z^{\cdot}) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots \tag{1.2}$$

The Schwarz function, named after the German mathematician Hermann Schwarz, is a complex-valued function that maps the unit disk E in the complex planes to itself. It is known by $f(z) = \frac{-z}{1-z^2}$ where z is a complex number. Specifically, if f and g are analytic functions defined on some domain D, then $f \prec g$ if there lie other analytic functions h defined

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on D such as $f(z^{\cdot}) = g\left(h(z^{\cdot})\right)$ $\forall z$ in $D^{\cdot}$. Thomas [1], and Pommerenke [2] defined the Hankel determinant $H_{k^{\cdot}}(c)$, for positive integer $k^{\cdot}, c$ for the function in $S^{\cdot}$ In the form eq (1), as shown below:

$$H_k(c) = \begin{vmatrix} a^{\cdot}_{n} & a^{\cdot}_{n+1} & \cdots & a^{\cdot}_{n+q-1} \\ a^{\cdot}_{n+1} & a^{\cdot}_{n+2} & \cdots & a^{\cdot}_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a^{\cdot}_{n+q-1} & a^{\cdot}_{n+q-2} & \cdots & a^{\cdot}_{n+2q-2} \end{vmatrix} \tag{1.3}$$

For fixedly positive integers k and c the growths of $H_k(c)$ as $n^{\cdot} \rightarrow \infty^{\cdot}$ have been determined by Obradovic [3], In 2023, with a boundedness boundary. Ehrenborg [4] investigated the Hankel determinant for exponential polynomials. The Hankel determinant of differentiable orders is achieved for k, c differential rates. For instance, when $k = 2$ and $c = 1$ it is defined as,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \ \left|a_1 a_3 - a_2^2\right| \ a_1 = 1 \tag{1.4}$$

The Fekete-Szego inequality is a well-known result in complex analysis and potential theory that provides an estimate for the growth of the Taylor coefficients of function that are analytical unit disk E. More precisely, let $f(z)$ be functions that are analytical in the open unit disc $|z^{\cdot}| < 1$, and let its Taylor series expansions be given by (1).In 2023, Singh et al. [5] and Fekete-Szego defined an inequality for the coefficient of a univalent analytic function on the unit disk. The Fekete-Szego inequality for some normalized analytic functions was investigated by various researchers working in the field of Geometric Functions Theory like Choi et al. [6], Ali et al. [7, 8], Owa and Cho [9, 10], Orhan and Cotirla [11], and Murugusundaramoorthy et al. [12]. In 2006, Shanmugam et al. [13] introduced The Fekete-Szego problem that can be applied to sub-classes of star-like functions when considering symmetrical points. Now for $k = 2, c = 2$ it can be obtained as,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}, \ \left|a_2 a_4 - a_3^2\right| \tag{1.5}$$

In 2012, Krishna and Ramreddy [14] introduced the 2nd Hankel determinant of means the univalent function, which is discussed here. Using a indeed close p, valent function the growth rate of the 2nd Hankel determinant was calculated by Shrigan [15] in 2022. Other researchers like Janteng et al. [16, 17], Bansal [18], Lee et al. [19], Lei et al. [20], Rain et al. [21], Rajya et al. [22], Zaprawa [23] introduced the coefficient of the function f that belong to the sub-class S of univalent function or to its sub-classes the upper-bound of the Hankel determinant for $k^{\cdot} = 2, c^{\cdot} = 3$ such as, $H_2(3)$ is defined as:

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}, \ \left| a_3 a_5 - a_4{}^2 \right| \tag{1.6}$$

For $k = 3$, $c = 1$ the Hankel Determinant, $H_3(1)$ is known as the 3rd Hankel Determinant we have

$$H^{\cdot}{}_3(1^{\cdot}) = \begin{vmatrix} a^{\cdot}{}_1 & a^{\cdot}{}_2 & a^{\cdot}{}_3 \\ a^{\cdot}{}_2 & a^{\cdot}{}_3 & a^{\cdot}{}_4 \\ a^{\cdot}{}_3 & a^{\cdot}{}_4 & a^{\cdot}{}_5 \end{vmatrix}$$

$$a^{\cdot}{}_3(a^{\cdot}{}_2 a^{\cdot}{}_4 - a^{\cdot}{}_3{}^2) - a^{\cdot}{}_4(a^{\cdot}{}_1 a^{\cdot}{}_4 - a^{\cdot}{}_2 a^{\cdot}{}_3) + a^{\cdot}{}_5(a^{\cdot}{}_3 - a^{\cdot}{}_2{}^2), \quad a^{\cdot}{}_1 = 1 \tag{1.7}$$

1.1. Applications

In mathematical physics, analytic functions are crucial for solving two-dimensional issues. Displacements and stresses in antiplane or in-plane fracture problems can be expressed as functions of complex potentials. Hankel matrices are created when an underlying state-space model or hidden Markov model is desired as a given sequence of output data. The A, B, and C matrices that characterize the state-space realization can be computed using the singular value decomposition of the Hankel matrix. The breakdown of non-stationary signals and time-frequency representation has been proven to be advantageous when using the Hankel matrix created from the signal. To get the weight parameters for the polynomial distribution approximation, the method of moments applied to polynomial distributions produces a Hankel matrix that must be inverted [24].

Calculus uses course equivalent q-calculus which is based on the solution of logical q-analogous outcomes without the use of limits. The systematic introduction of q-calculus is credited to Lashin [25]. Khanetal

[26] introduced and presented definitions for q-derivative. q-derivative of functions f be a normalized-analytic function is known as

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq E,$$

and $D_q f(0) = f'(0)$ where $q \in (0, 1)$ taking $q \to 1^-$ we get $D_q f \to f'$. In 2007, Babalola [27] was defined as the 1st person to analyze the upper-bound of the 3rd Hankel determinant for sub-classes of S. Other researchers like Vamshee Krishna et al. [28], Patil and Khairnar [29], Prajapat et al. [30], Yalcin and Altinkaya [31], Cho et al. [32], Lecko et al. [33]. Kowalczyk et al. [34], Mohd Narzan et al. [35], Several other researchers like Mendiratta et al. [36], Haiyan Zhang et al. [37], khan et al. [38], and Senguttuvan et al. [39] defined A thorough sub-class of analytic functions with respect to the symmetrical point that has been developed. The current study is expanded by using quantum calculus and tends to investigate the upper bounds of the 3rd Hankel Determinant, for the classes of a star-like function with respect to symmetrical points subordinate to exponential functions. Mahmood et al. [40] Shi et al. [41], Verma et al. [42], Viswanadh et al. [43], Omer [44], Joshi et al. [45], Breaz et al [46], Wang [47], and investigated the class of univalent function star-like with respect to symmetrical points. Here, the following subclass of starlike function are defined below:

Definition 1.1. A function $f \in A$ and f is known to be in the class $S_s^*(e^{qz})$ as

$$\frac{2[zf'(z)]}{f(z)-f(-z)} \prec e^{qz}, \ z \in E, \tag{1.8}$$

we note that taking $q \to 1^-$ in the above definition, we obtain the known class $S_s^*(e^z)$ see [48]. The following Lemmas are required to demonstrate the intended outcomes.

Lemma 1.1. [49] If $p \in P$, then $|p_n| \leq 2, \forall n \in N$.

Lemma 1.2. [50] If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is such that $Re(p(z)) > 0$ in E, then for some x, z with $|x| \leq 1, |z| \leq 1$, we have

$$2p_2 = p_1^2 + x(4 - p_1^2), \text{ for some } x, |x| \leq 1 \tag{1.9}$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \tag{1.10}$$

Lemma 1.3. [51] If $p \in P, then\ |p_2 - vp_1^2| \leq \max\ |\{1, |2v^{\cdot} - 1^{\cdot}|\}$ for any $v^{\cdot} \in C^{\cdot}$.

3. MAIN RESULTS

Theorem.3.1: If $f \in S_s^*(e^{qz})$ then $|a_2| \leq \frac{q}{2}, |a_3| \leq \frac{q}{2}, |a_4| \leq \left|\frac{q}{4} + \left(\frac{-4q+3q^2}{8}\right) + \left(\frac{12q-18q^2+5q^3}{48}\right)\right|, |a_5| \leq \left|\frac{q}{4} + \left(\frac{-2q+q^2}{4}\right) + \left(\frac{-6q+9q^2-3q^3+q^4}{24}\right)\right|$.

Proof: As $f \in S_s^*(e^{qz})$ as

$$\frac{2[zf'(z)]}{f(z^{\cdot}) - f(-z^{\cdot})} = e^{qw(z)} \ . \tag{2.1}$$

Using **Eq.** (**2.1**), we consider

$$\frac{2[zf'(z)]}{f(z) - f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 z^4 - 2a_3 a_2) z^3 + (4a_5 z^5 - 2a^2{}_3) z^4 + \ldots \tag{2.2}$$

Let us define the function,

$$p(z^{\cdot}) = \frac{1 + qw(z^{\cdot})}{1 - qw(z^{\cdot})}.$$

Equivalent,

$$q^{\cdot}w(z^{\cdot}) = \frac{p(z^{\cdot})-1}{p(z^{\cdot})+1}, \tag{2.3}$$

Consider

$$qw(z) = \frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} - \frac{p_1 p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}\right) z^4 \ldots \tag{2.4}$$

Since we have

$$e^{qw^{\cdot}(z)} = 1 + qw(z^{\cdot}) + \frac{\left(qw(z^{\cdot})\right)^2}{2!} + \frac{\left(qw(z^{\cdot})\right)^3}{3!} + \frac{\left(qw(z^{\cdot})\right)^4}{4!} + \cdots \tag{2.5}$$

We get

$$e^{qw(z)} = 1 + \frac{qp_1z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)q + \left(\frac{p_1^2q^2}{8}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48}\right)q + \left(\frac{-p_1^2q^2}{8} + \frac{p_1p_2q^2}{4} + \frac{p_1^3q^3}{48}\right)z^3 + \left(\frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_1^2p_2}{16} + \frac{p_1^4}{384}\right)q + \left(\frac{p_2^2}{8} + \frac{3p_1^4}{32} - \frac{p_1^2p_2}{4} + \frac{p_1p_3}{4}\right)q^2 + \left(-\frac{p_1^2p_2}{16} - \frac{p_1^4}{32}\right)q^3 + \left(\frac{p_1^4q^4}{384}\right)z^4 + \cdots \tag{2.6}$$

From **Eq. (2.2)** and **Eq. (2.6)**, we compare the coefficient, and we get

$$a_2 \le \frac{qp_1}{4}, a_3 = \left(\frac{p_2}{4} - \frac{p_1^2}{8}\right)q + \frac{q^2p_1^2}{16},$$

$$a_4 = \frac{p_3q}{8} - \frac{p_1p_2q}{8} + \frac{3p_1p_2q^2}{32} + \frac{p_1^3q}{32} - \frac{p_1^3q^2}{64} + \frac{5p_1^3q^3}{384},$$

$$a_5 = (\frac{p_4q}{8} + \left(-\frac{p_1p_3q}{8} + \frac{p_1p_3q^2}{16}\right) + \left(\frac{3p_1^2p_2q}{32} - \frac{3p_1^2p_2}{32} - \frac{3p_1^2p_2q^2}{32}\right) + \left(\frac{p_1^4q^2}{128} - \frac{p_1^4q}{64} - \frac{p_1^4q^3}{128} + \frac{p_1^4q^4}{384}\right). \tag{2.7}$$

By using **Lemma 1.1** and **Lemma 1.3** in **Eq. (2.7)**, we get

$$|a_2| \le \frac{q}{2}, |a_3| \le \frac{q}{2}, |a_4| \le \left|\frac{q}{4} + \left(\frac{-4q + 3q^2}{8}\right) + \left(\frac{12q - 18q^2 + 5q^3}{48}\right)\right|,$$

$$|a_5| \le \left|\frac{q}{4} + \left(\frac{-2q+q^2}{4}\right) + \left(\frac{-6q+9q^2-3q^3+q^4}{24}\right)\right|. \tag{2.8}$$

which are the required results.

Theorem 3.2: If $f \in S_s^*(e^{qz})$ then $|a_3 - a_2^2| \le \frac{q}{2}$

Proof: From **Eq. (2.7)** in **Theorem 3.1** we have

$$a_2 \le \frac{qp_1}{4}, a_3 = \left(\frac{p_2}{4} - \frac{p_1^2}{8}\right)q + \frac{q^2p_1^2}{16}. \tag{2.9}$$

On simplifying, we get

$$|a_3 - a_2^2| = \left|\frac{qp_2}{4} - \frac{q^2p_1^2}{8}\right|. \tag{2.10}$$

By using **Lemma 1.3**, we get

$$|a_3 - a_2^2| \le \frac{q}{2}. \tag{2.11}$$

which are the required results.

Theorem 3.3: If $f \in S_s^*(e^{qz})$ then $|a_2a_3 - a_4| \leq$
$\frac{1}{48(q^2-6q-12)}(3q(3q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^4 -$
$36q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q^3 +$
$6q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^4 + 12q^5 +$
$2q^4\sqrt{-2q^3+8q^2+48q+64} + 36q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q^2 -$
$72q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^3 - 144q^4 -$
$8\sqrt{-2q^3+8q^2+48q+64}\,q^3 + 432q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q +$
$72q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^2 + 144q^3 -$
$48\sqrt{-2q^3+8q^2+48q+64}\,q^2 + 432q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 +$
$864q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q + 1728q^2 -$
$16\sqrt{-2q^3+8q^2+48q+64}\,q + 864q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right) +$
$1600q.$

Proof: From **Eq.** (**2.7**) of **Theorem 3.1**, we have

$$|a_2a_3 - a_4| = \left|\frac{q^2p_1p_2}{16} - \frac{p_1^3q^2}{32} + \frac{q^3p_1^3}{64} - \frac{p_3q}{8} + \frac{p_1p_2q}{8} - \frac{3p_1p_2q^2}{32} - \frac{p_1^3q^2}{32} + \frac{p_1^3q^2}{64} - \frac{5p_1^3q^3}{384}\right|. \tag{2.12}$$

Denotes $|x| = t \in [0,1], p_1 = e \in [0,2]$, using triangle inequality (**2.12**) we have

$$|a_2a_3 - a_4| \leq \frac{e(4-e^2)qt^2}{32} - \frac{t(4-e^2)q^2e}{64} - \frac{q(4-e^2)}{16} + \frac{e^3q^3}{384},$$

suppose that

$$F(e,1) \equiv \frac{e(4-e^2)qt^2}{32} - \frac{t(4-e^2)q^2e}{64} - \frac{q(4-e^2)}{16} + \frac{e^3q^3}{384}.$$

Thus, we get $\frac{\partial F}{\partial t} = \frac{(4-e^2)etq}{16} - \frac{(4-e^2)eq^2}{64} \geq 0$, the function *F(e,t)* is non-decreasing for any *t* in *[0,1]*.this show that *F(e,t)* has max value at *t=1.*

$$MaxF(e,t) = F(e,1) = \frac{e(4-e^2)qt^2}{32} - \frac{t(4-e^2)q^2e}{64} - \frac{q(4-e^2)}{16} + \frac{e^3q^3}{384},$$

which implies that

$$M(e) = \frac{e(4-e^2)qt^2}{32} - \frac{t(4-e^2)q^2e}{64} - \frac{q(4-e^2)}{16} + \frac{e^3q^3}{384},$$

Then,

$$M'(e) = \left(-\frac{qe^2}{16} + \frac{(-e^2+4)q}{32} - \frac{e^2q^2}{32} - \frac{(-e^2+4)q^2}{64} - \frac{eq}{8} + \frac{e^2q^3}{128}\right),$$

$$M'(e) = \text{be lost } \ e = m^* = -\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}.$$

A simple computational yield that $M''(e) < 0$, which means that the functions $M^{\cdot}(e)$ can take a max value at $m^* = -\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64})}{q^2-6q-12}$.

Hence we get

$$|a_2a_3 - a_4| \leq \frac{1}{48(q^2-6q-12)}(3q(3q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^4 -$$
$$36q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q^3 +$$
$$6q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^4 + 12q^5 +$$
$$2q^4\sqrt{-2q^3+8q^2+48q+64} + 36q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q^2 -$$
$$72q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^3 - 144q^4 -$$
$$8\sqrt{-2q^3+8q^2+48q+64}\,q^3 + 432q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 q +$$
$$72q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q^2 + 144q^3 -$$
$$48\sqrt{-2q^3+8q^2+48q+64}\,q^2 + 432q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)^2 +$$
$$864q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64}}{q^2-6q-12}\right)q + 1728q^2 -$$

$16\sqrt{-2q^3+8q^2+48q+64}\,q+864q\left(-\frac{2(-4+\sqrt{-2q^3+8q^2+48q+64})}{q^2-6q-12}\right)+$
$1600q.$ (2.13)

which are the required results.

Theorem 3.4: If $f\in S_s^*(e^{qz})$ then $|a_2a_4-a_3^2|\le\frac{3q^2}{8}$

Proof: From **Eq.** (**2.7**) of **Theorem 3.4**, we have

$$|a_2a_4-a_3^2|=\left|\frac{p_1p_3q^2}{32}+\frac{3p_1^2p_3q^3}{128}-\frac{p_1^2p_3q^2}{32}+\frac{p_1^4q^2}{128}-\frac{3p_1^4q^3}{256}+\frac{5p_1^4q^4}{1536}-\frac{p_1^2p_2}{128}-\frac{p_1^4}{1536}-\frac{p_2^2q^2}{16}-\frac{p_2p_1^2q^3}{32}+\frac{p_2p_1^2q^2}{16}\right|,$$

we use the **Lemma 1.3** we have

$$|a_2a_4-a_3^2|=\left|\frac{p_1q^2(4-p_1^2)(1-|x|^2)z}{64}-\frac{p_1^2(4-p_2^2)x^2q^2}{128}+\frac{3p_1^2q^3x(4-p_1^2)}{256}-\frac{x^2q^2(4-p_1^2)^2}{64}+\frac{5p_1^4q^4}{1536}-\frac{p_1^4q^2}{64}+\frac{p_1^4q^2}{32}-\frac{p_1^4q^4}{64}-\frac{p_1^4q^4}{256}\right|. \quad (2.14)$$

Denotes $|x|=t\in[0,1], p_1=e\in[0,2]$ then using tri-angle inequality we have

$$|a_2a_4-a_3^2|\le\frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right), \quad (2.15)$$

which implies that

$$F(e,t)=\frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right). \quad (2.16)$$

Thus, we get

$$\frac{\partial F}{\partial t}=\frac{3(4-e^2)e^2q^2}{256}+\frac{(4-e^2)e^2q^2}{64}+\frac{(4-e^2)^2q^2t}{32}+\frac{(4-e^2)e^2q^2t}{64}\ge 0,$$

which gives that *F(e, t)* is increasing for any then *t* in *[0,1]*, this show that *F (e, t)* has a maxi value at *t = 1*.

$$Max\ F(e^{\cdot},t) = F(e^{\cdot},t) = \frac{q^2(4-e^{\cdot 2})}{32} + \frac{e^{\cdot 2}(4-e^{\cdot 2})t^2q^2}{128} + \frac{3e^{\cdot 2}q^3t(4-e^{\cdot 2})}{256} + \frac{t^2q^2(4-e^{\cdot 2})^2}{64} + \left(\frac{5e^{\cdot 4}q^4}{1536} - \frac{e^{\cdot 4}q^2}{64} + \frac{e^{\cdot 4}q^2}{32} - \frac{e^{\cdot 4}q^4}{64} - \frac{e^{\cdot 4}q^4}{256}\right). \quad (2.17)$$

Let us define

$$M(c) = \frac{q^2(4-e^2)}{32} + \frac{e^2(4-e^2)t^2q^2}{128} + \frac{3e^2q^3t(4-e^2)}{256} + \frac{t^2q^2(4-e^2)^2}{64} + (\frac{5e^4q^4}{1536} - \frac{e^4q^2}{64} + \frac{e^4q^2}{32} - \frac{e^4q^4}{64} - \frac{e^4q^4}{256}). \quad (2.18)$$

We have

$$M'(e) = \frac{5e^3q^2}{128} - \frac{(-5e^2+4)eq^2}{128} - \frac{(eq^2)e}{16} - \frac{25e^3q^4}{384},$$

$M'(e)$ vanishes at $e = 0$. A simple compilation yield that $M''(e) < 0$, which mean that the functions $M(e)$ has max values at $e = 0$. We get

$$|a_2a_4 - a_3^2| \leq M(0) = \frac{3q^2}{8}\ , \quad (2.19)$$

which are the required results.

Theorem 3.5: If $f \in S_s^*(e^{qz})$ then $|H_3(1) =$

$$\frac{1}{2304(q^2-6q-12)^2}(q^3\left(15q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2 q^4 - 180q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2 q^3 + 30q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^4 + 10q^4\sqrt{-2q^3+8q^2+48q+64} + 60q^5 + 180q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2 q^2 - 360q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^4 - 40\sqrt{-2q^3+8q^2+48q+64}\,q^3 - 288q^4 + 2160q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2 q +$$

$$360q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^2-$$
$$240\sqrt{-2q^3+8q^2+48q+64}\,q^2-4464q^3+$$
$$2160q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2+$$
$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q-$$
$$80\sqrt{-2q^3+8q^2+48q+64}\,q+13824q^2+$$
$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)+702008q+62208)\Bigg)+$$
$$\frac{1}{2}\left(\frac{1}{4}q-\frac{1}{8}q^2-\frac{1}{8}q^3+\frac{1}{24}q^4\right)q.$$

Proof: $H'_3(1)=a'_3(a'_2a'_4-a'^2_3)-a'_4(a_1a_4-a'_2a'_3)+a'_5(a'_3-a'^{2.}_2)$, $a_1=1$

(2.20)

By applying triangle inequality, we get

Now, substituting the **Eq.** (**2.8**),(**2.11**),(**2.13**),(**2.19**), in (**2.20**) we get

$$|H_3(1)|\le\frac{1}{2304(q^2-6q-12)^2}\Bigg(q^3\Bigg(15q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2q^4-$$
$$180q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2q^3+$$
$$30q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^4+$$
$$10q^4\sqrt{-2q^3+8q^2+48q+64}+60q^5+$$
$$180q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2q^2-$$
$$360q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^4-$$
$$40\sqrt{-2q^3+8q^2+48q+64}\,q^3-288q^4+$$
$$2160q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2q+$$

$$360q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q^2 -$$
$$240\sqrt{-2q^3+8q^2+48q+64}\,q^2 - 4464q^3 +$$
$$2160q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)^2 +$$
$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right)q -$$
$$80\sqrt{-2q^3+8q^2+48q+64}\,q + 13824q^2 +$$
$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3}+8q^2+48q+64\right)}{q^2-6q-12}\right) + 702008q + 62208)\Bigg) +$$
$$\frac{1}{2}\left(\frac{1}{4}q - \frac{1}{8}q^2 - \frac{1}{8}q^3 + \frac{1}{24}q^4\right)q \ , \qquad (2.21)$$

which are the required results.

4. DISCUSSION

The coefficients of the univalent functions attracted the attention of researchers working in the field of Geometric Functions Theory. On the other hand, Quantum Calculus and Quantum theory have been applied to the concepts of Geometric Functions Theory to advance the known classes and results. The Fekete-Szego problem are a well-known coefficient problem in this field which has a variety of application in other sciences. The well-known application of the Fekete problem is the Hankel determinant. The Hankel determinant of order two have been investigated by various researchers in this field but the 3rd order Hankel determinant for starlike functions with respect to symmetric points have attained the attention of researcher nowadays. To fill this research gap we have worked on the 3rd order Hankel determinant for our new class of starlike functions with respect to symmetric points subordination q-exponential function. Where the concepts of quantum calculus and q-derivative operator have been applied. The current study determined that on taking $q \to 1^-$ for the result proved in this research article, similar results were obtained that were already proved in [47].

4.1. Conclusion

The class of starlike functions with respect to symmetric points and the q-extension of these class has been investigated. Conclusively, a new class of starlike functions with respect to symmetric points associated with q-exponential function by using the subordination technique is defined and studied in this research. Some remarkable results, including coefficient inequalities, the Fekete Szegö problem, and the third-order Hankel determinant have been investigated. It was indicated that the new class along with the associated main theorems are the advancement of the results and classes, which has already been studied by the researcher working in the field of Geometry Functions Theory (GFT).

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