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Novel/Old Generalized Multiplicative Zagreb Indices of Some Special Graphs

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ABSTRACT

Topological descriptor is a fixed real number directly attached with the molecular graph to predict the physical and chemical properties of the chemical compound. Gutman and Trinajstic' elaborated the first Zagreb index (ZI) which was based on degree in 1972. Ali and Trinajstic' defined a connection number (CN) based topological descriptor in 2018. They found that the CN-based Zagreb indices have a greater chemical capability for thirteen physicochemical properties of octane isomers.For $\kappa,\lambda \in R-\{0\}$, the generalized ZI and the generalized first Zagreb connection index (ZCI) of a graph Q is $Z_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[\mathrm{d(l)}^{\kappa} d(m)^{\lambda}+\mathrm{d(m)}^{\kappa} d(l)^{\lambda}]$ and $C_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[\tau(\mathrm{l})^{\kappa} \tau(m)^{\lambda}+\tau(\mathrm{m})^{\kappa} \tau(l)^{\lambda}]$, where $d_Q(m)$ and $\tau_Q(m)$ are the degree and CN of the vertex m in Q. In this paper, the generalized first, second, third, and fourth multiplicative ZCIs are defined. Some exact solutions are also developed for the generalized multiplicative ZI and the above-mentioned generalized multiplicative versions of some special graphs, which are flower, sunflower, wheel, helm, and gear. The results $(MC^2_{\kappa,\lambda}, MC^3_{\kappa,\lambda}$ and $MC^4_{\kappa,\lambda})$ are the generalized forms of the results $(MZC^*_1, MZC^*_2$ and $MZC^*_3)$ where, $\kappa,\lambda=1,$ respectively.

Keywords: connection number, degree, generalized multiplicative Zagreb indices, special graphs

1. INTRODUCTION

Recently, graph theory has provided some useful tools for the study of different structures. These tools are known by the name of topological indices (TIs) and used mostly in pharmaceutical industry, medicinal fields, and the study of crystalline and nano materials, see [1-3]. Additionally, TIs

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are used in quantitative-structure-activity relationships (QSARs) and quantitative-structure-property relationships (QSPRs). These relations have been joined to molecular-structures with their biological-structures [4, 5]. Mathematically, TIs can be defined as $\xi = \phi(\mathrm{M})$, where ϕ is a real-valued function, M is a molecular-structure, and ξ is a real value which depends on M.

Multiplicative Zagreb indices (ZIs) have outstanding importance in varius research fields. They play a significant role in analytical chemistry, toxicology, medicine, physical chemistry, pharmaceutical research, material sciences, environmental chemistry, and engineering. They can be joinedwith several properties of chemical compounds [6]. Todeshine et al. [7] and Eliasi et al. [8] separately developed multiplicative ZIs. In 2011, Gutman [9] used multiplicative ZIs to compute results of different tree graphs. Xu and Hua [10] used these multiplicative versions with the nameof multiplicative sum ZI to find different results of trees, as well as unicyclic and bicyclic graphs. Xu and Das [11] also used multiplicative sum ZI to find different results of trees, as well as unicyclic and bicyclic graphs, by using another method shorter than [8]. Multiplicative ZIs and multiplicative sum ZIs are being rapidly used in theforthcoming research. For more information, see [12–18].

In 2011, Azari [19] defined the generalized ZI to compute nanotubes and nanotori networks. He also defined graphical products, such as Cartesian product, sum, lexicographic product, union, corona product, disjunction, strong product, and symmetric difference with the help of generalized ZI [20]. Farahani and Kanna [21] computed generalized ZI for V-phenylenic nanotori and nanotubes. Sarkar et al. [22] studied the generalized ZI of several allotropes of carbon, such as carbon graphite, crystal cubicstructure of carbon, and graphene. Sarkar et al. [23] used the generalized ZI findings to compute different results of regular dendrimers. Sarkar et al. [24] used this generalized ZI with another name, that is, (a,b)-Zagreb index to compute several derived networks. Mirajkar et al. [25] computed the generalized ZI of capra operation of cycle C_v on v-vertces. Wang et al. [26] used both generalized and generalization ZIs to compute several results of silicon-carbon graphs.

Degree-based TIs have been classified into degree and connection number (CN) [27]. Degree and CN indicate the number of vertices whose distance from vertex m must be 1 and 2, respectively. These indices have a powerful role in the study of chemical compounds and biological experiments [28, 29]. The first degree- based TI came into existence from the study of ψ-electron energy. This TI was revolutionized by Gutman and Trinajstic' [30] with the name of ZI. They also shortly worked on another TI named as CN TI which is the number of those vertices whose distance from a particular vertex is 2. As such, TIs are based on CN [31]. Currently, these CN-based ZIs are used to investigate the physicochemical properties of chemical compounds, such as their stability, boiling point, strain energy, acentric factor, and entropy more than classical ZIs, see [32–34]. Ye et al. [35] computed ZCIs for nanotubes. Wang et al. [36] computed ZCIs of k-dimensional Benes networks. Hussain et al. [37] computed ZCIs of subdivided graph on $\vartheta^{\phi}\chi$. For more details related to CN-based ZIs or leap ZIs, the reader may see [38–47].

In this paper, the generalized first, second, third, and fourth multiplicative ZCIs are defined. Exact solutions for the old and novel generalized multiplicative ZIs of some special graphs including wheel, gear, helm, flower, and sunflower are also developed. The rest of the paper is structured as follows: Section II gives the basic definitions of the degree- and CN-based ZIs, Section III provides the definitions of several special graphs and their main results which are related to generalized multiplicative ZIs and generalized multiplicative ZCIs, while section IV presents the conclusion.

2. NOTATIONS AND PRELIMINARIES

Let Q be a connected and simple graph. $|E(Q)| = s$ and $|V(Q)| = v$ are the size and order of graph Q, respectively. The degree $(d_Q(m))$ of vertex m is the number of incident edges. If we sum the degrees of all the vertices adjacent to m, it is known as degree sum of a vertex. Mathematically, it is written as $DS(m) = \sum_{l \in N_Q(m)} d_Q(l)$. If we multiply the degrees of all the vertices adjacent to m, it is known as degree product of a vertex. Mathematically, it is written as

$DP(m)=\prod_{l\in N_Q(m)} d_Q(l)$. If $N_\delta(m)$ is the δ-neighborhood of a vetex m, then $|N_\delta(m)=d_\delta(m)|$ (number of δ-neighbors of a vertex m). If we put $\delta=2$, it yields $d_2(m)=\tau(m)$ (CN of a vertex m). Furthermore, $\bar{d}_Q(m)=\frac{\sum_{m\in V(Q)} d_Q(m)}{v}$ and $\bar{\tau}_Q(m)=\frac{\sum_{m\in V(Q)} \tau_Q(m)}{v}$ are the average degree and average CN of order v. The computed value of $\bar{d}_Q(m)$ and $\bar{\tau}_Q(m)$ are $\frac{2s}{v}$ and $\frac{M_1-2s}{v}$, respectively. For more terminologies and notions [48].

Definition 2.1 [42]. Let Q be a connected graph and $\forall\kappa,\lambda\in R-\{0\}$. The δ-distance reneralized Zagreb index $(Z^{\delta}_{\kappa,\lambda}(Q))$ is

$$Z^{\delta}_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[d_\delta(l)^\kappa d_\delta(m)^\lambda + d_\delta(m)^\kappa d_\delta(l)^\lambda].$$

where $\delta\geq 1$. If we put $\delta=1$, we get the generalized ZI as follows:

$$Z_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[d(l)^\kappa d(m)^\lambda + d(m)^\kappa d(l)^\lambda].$$

Azari and Iranmanesh [19] (2011) defined this degree-based generalized version. If we replace sum by product, we obtain the generalized multiplicative ZI as follows:

$$MZ_{\kappa,\lambda}(Q)=\prod_{lm\in E(Q)}[d(l)^\kappa d(m)^\lambda + d(m)^\kappa d(l)^\lambda].$$

If we put $\delta=2$, we get generalized ZCIs based on connection number $(\tau=d_2)$ as follows:

Definition 2.2 [42]. Let Q be a connected graph and $\forall\kappa,\lambda\in R-\{0\}$. The generalized first, second, third, and fourth ZCIs are as follows:

$$C_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[\tau(l)^\kappa \tau(m)^\lambda + \tau(m)^\kappa \tau(l)^\lambda],$$

$$C^{2}_{\kappa,\lambda}(Q)=\sum_{lm\in E(Q)}[d(l)^\kappa \tau(m)^\lambda + d(m)^\kappa \tau(l)^\lambda],$$

$$C^3_{\kappa,\lambda}(Q) = \sum_{\mathrm{lm}\in \mathrm{E(Q)}}[d(l)^\kappa \tau(l)^\lambda + d(m)^\kappa \tau(m)^\lambda],$$

$$C^4_{\kappa,\lambda}(Q) = \sum_{\mathrm{lm}\in \mathrm{E(Q)}}[d(l)^\kappa \tau(l)^\lambda \times d(m)^\kappa \tau(m)^\lambda].$$

If we change sum into product, we get generalized multiplicative ZCIs as follows.

Definition 2.3. Let Q be a connected graph and $\forall \kappa, \lambda \in R-\{0\}$. The generalized first, second, third, and fourth multiplicative ZCIs are as follows:

$$MC_{\kappa,\lambda}(Q) = \prod_{lm\in E(Q)}[\tau(l)^\kappa \tau(m)^\lambda + \tau(m)^\kappa \tau(l)^\lambda],$$

$$MC^2_{\kappa,\lambda}(Q) = \prod_{lm\in E(Q)}[d(l)^\kappa \tau(m)^\lambda + d(m)^\kappa \tau(l)^\lambda],$$

$$MC^3_{\kappa,\lambda}(Q) = \prod_{lm\in E(Q)}[d(l)^\kappa \tau(l)^\lambda + d(m)^\kappa \tau(m)^\lambda],$$

$$MC^4_{\kappa,\lambda}(Q) = \prod_{lm\in E(Q)}[d(l)^\kappa \tau(l)^\lambda \times d(m)^\kappa \tau(m)^\lambda].$$

3.DEFINITIONS AND MAIN RESULTS

This section presents the definitons of some special graphs, such as wheel, gear, helm, flower, and sunflower graphs, respectively. Also, main results for the generalized multiplicative ZI and generalized multiplicative ZCIs are presented.

3.1. Wheel Graph W_v

The wheel graph W_v is defined by joining K_1 and C_v, where K_1 and C_v are the complete and cyclic graphs of orders 1 and v, respectively. Thus, $v+1$ and $2v$ are the order $(|V(W_v)|)$ and $(|E(W_v)|)$ of wheel graph, respectively. Apex is the vertex corresponding to K_1 and rim is the set of the vertices corresponding to C_v. Figure 1 depicts the graphical representation of wheel graph.

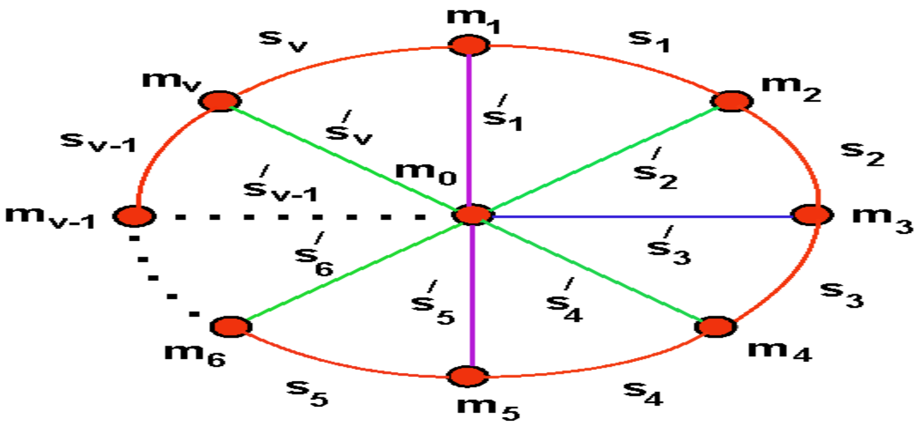

Figure 1. Wheel Graph W_v

Table 1. Partition of Vertices for W_v

Degree/CN	Apex (m_0)	Rim (m_i)
d_{W_v}	v	3
τ_{W_v}	0	$v-3$

Theorem 3.1. Let W_v be a wheel graph with order $v+1$. The generalized multiplicative ZI and the generalized first, second, third, and fourth multiplicative ZCIs of W_v are as follows:

1. $MZ_{\kappa,\lambda}(W_v) = 2^{\mathrm{r}} 3^{(\kappa+\lambda)\mathrm{v}} \times (r^{\kappa} 3^{\lambda} + 3^{\kappa} r^{\lambda})^{v}$,

2. $MC_{\kappa,\lambda}(W_v) = 0$,

3. $MC^{2}_{\kappa,\lambda}(W_v) = 2^{\mathrm{v}} (3v)^{\kappa v} (v-3)^{2\lambda v}$,

4. $MC^{3}_{\kappa,\lambda}(W_v) = 2^{\mathrm{v}} (9)^{\kappa v} (v-3)^{2\lambda v}$,

5. $MC^{4}_{\kappa,\lambda}(W_v) = 0$.

Proof: 1 Since W_v is a wheel graph of order $v+1$, where $v \geq 4$. Also, m_0 is the apex vertex and $\{m_1, m_2, \ldots\ldots, m_v\}$ is the set of rim vertices for W_v. Then, by definition,

$$MZ_{\kappa,\lambda}(W_v) = \prod_{lm\in E(W_v)} \left[d_{W_v}(l)^{\kappa} d_{W_v}(m)^{\lambda} + d_{W_v}(m)^{\kappa} d_{W_v}(l)^{\lambda}\right]$$

$$= \prod_{m_0 m_i \in E(W_v)} \left[d_{W_v}(m_0)^{\kappa} d_{W_v}(m_i)^{\lambda} + d_{W_v}(m_i)^{\kappa} d_{W_v}(m_0)^{\lambda}\right] \times \prod_{m_i m_{i+1} \in E(W_v)} \left[d_{W_v}(m_i)^{\kappa} d_{W_v}(m_{i+1})^{\lambda} + d_{W_v}(m_{i+1})^{\kappa} d_{W_v}(m_i)^{\lambda}\right].$$

By using Table 1, we get

$$= [\mathrm{v}^{\kappa} 3^{\lambda} + 3^{\kappa} v^{\lambda}]^{v} \times [3^{\kappa} 3^{\lambda} + 3^{\kappa} 3^{\lambda}]^{v}$$

$$= 2^{\mathrm{v}} 3^{(\kappa+\lambda)\mathrm{v}} \times (\mathrm{v}^{\kappa} 3^{\lambda} + 3^{\kappa} v^{\lambda})^{v}.$$

2. $MC_{\kappa,\lambda}(W_v) = \prod_{lm\in E(W_v)} \left[\tau_{W_v}(l)^{\kappa} \tau_{W_v}(m)^{\lambda} + \tau_{W_v}(m)^{\kappa} \tau_{W_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(W_v)} \left[\tau_{W_v}(m_0)^{\kappa} \tau_{W_v}(m_i)^{\lambda} + \tau_{W_v}(m_i)^{\kappa} \tau_{W_v}(m_0)^{\lambda}\right] \times \prod_{m_i m_{i+1} \in E(W_v)} \left[\tau_{W_v}(m_i)^{\kappa} \tau_{W_v}(m_{i+1})^{\lambda} + \tau_{W_v}(m_{i+1})^{\kappa} \tau_{W_v}(m_i)^{\lambda}\right]$$

$$= [0^{\kappa}(v-3)^{\lambda} + (\mathrm{v}-3)^{\kappa} 0^{\lambda}]^{v} \times [(\mathrm{v}-3)^{\kappa}(v-3)^{\lambda} + (\mathrm{v}-3)^{\kappa}(v-3)^{\lambda}]^{v}$$

$$= 0.$$

3. $MC^{2}_{\kappa,\lambda}(W_v) = \prod_{lm\in E(W_v)} \left[d_{W_v}(l)^{\kappa} \tau_{W_v}(m)^{\lambda} + d_{W_v}(m)^{\kappa} \tau_{W_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(W_v)} \left[d_{W_v}(m_0)^{\kappa} \tau_{W_v}(m_i)^{\lambda} + d_{W_v}(m_i)^{\kappa} \tau_{W_v}(m_0)^{\lambda}\right] \times \prod_{m_i m_{i+1} \in E(W_v)} \left[d_{W_v}(m_i)^{\kappa} \tau_{W_v}(m_{i+1})^{\lambda} + d_{W_v}(m_{i+1})^{\kappa} \tau_{W_v}(m_i)^{\lambda}\right]$$

$$= [\mathrm{r}^{\kappa}(r-3)^{\lambda} + 3^{\kappa} 0^{\lambda}]^{r} \times [3^{\kappa}(r-3)^{\lambda} + 3^{\kappa}(r-3)^{\lambda}]^{r}$$

$$= 2^{\mathrm{v}}(3v)^{\kappa v}(v-3)^{2\lambda v}.$$

4. $MC^{3}_{\kappa,\lambda}(W_v) = \prod_{lm\in E(W_v)} \left[d_{W_v}(l)^{\kappa} \tau_{W_v}(l)^{\lambda} + \tau_{W_v}(m)^{\kappa} d_{W_v}(m)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(W_v)} \left[d_{W_v}(m_0)^{\kappa} \tau_{W_v}(m_0)^{\lambda} + d_{W_v}(m_i)^{\kappa} \tau_{W_v}(m_i)^{\lambda}\right] \times \prod_{m_i m_{i+1} \in E(W_v)} \left[d_{W_v}(m_i)^{\alpha} \tau_{W_v}(m_i)^{\beta} + d_{W_v}(m_{i+1})^{\alpha} \tau_{W_v}(m_{i+1})^{\beta}\right]$$

$$=[v^{\kappa}0^{\lambda}+3^{\kappa}(v-3)^{\lambda}]^{v} \;\times\; [3^{\kappa}(v-3)^{\lambda}+3^{\kappa}(v-3)^{\lambda}]^{v}$$

$$=2^{v}(9)^{\kappa v}(v-3)^{2\lambda v}.$$

5. $MC^{4}_{\alpha,\beta}(W_v) = \prod_{lm\in E(W_v)} \left[d_{W_v}(l)^{\kappa}\tau_{W_v}(l)^{\lambda}\times\tau_{W_v}(m)^{\kappa}d_{W_v}(m)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(W_v)}\left[d_{W_v}(m_0)^{\kappa}\tau_{W_v}(m_0)^{\lambda}\times d_{W_v}(m_i)^{\kappa}\tau_{W_v}(m_i)^{\lambda}\right]\times\prod_{m_im_{i+1}\in E(W_v)}$$
$$\left[d_{W_v}(m_i)^{\kappa}\tau_{W_v}(m_i)^{\lambda}\times d_{W_v}(m_{i+1})^{\kappa}\tau_{W_v}(m_{i+1})^{\lambda}\right]$$

$$=[v^{\kappa}0^{\lambda}\times 3^{\kappa}(v-3)^{\lambda}]^{v} \;\times\; [3^{\kappa}(v-3)^{\lambda}\times 3^{\kappa}(v-3)^{\lambda}]^{v}$$

$$=0.$$

3.2. Gear Graph G_v

The gear graph G_v is obtained from the wheel graph by including a new vertex between each pair of the adjacent vertices of rim. Thus, $2v+1$ and $3v$ are the order $(|V(G_v)|)$ and size $(|E(G_v)|)$ of gear graph, respectively. Also, the bipartite wheel graph is called gear graph. Figure 2 depicts the graphical representation of gear graph.

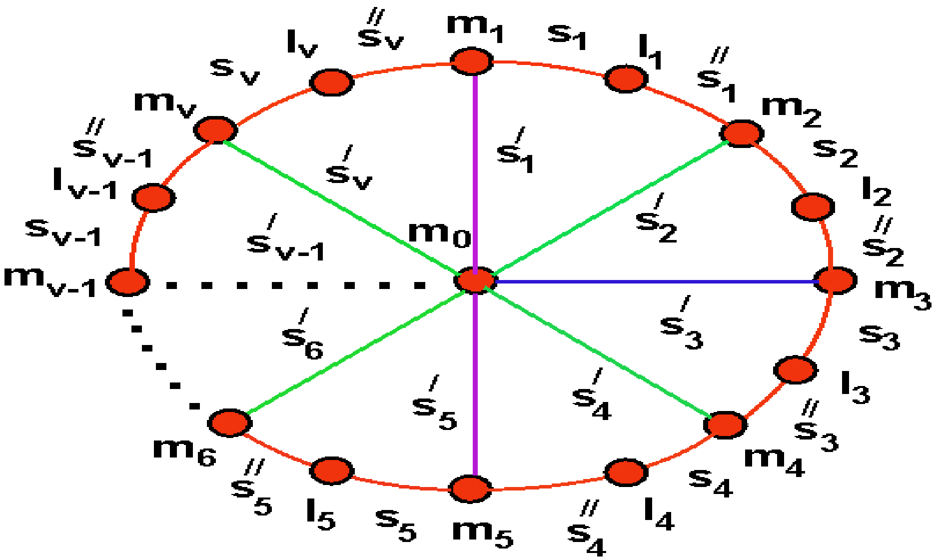

Figure 2. Gear Graph G_v

Table 2. Partition of Vertices for G_v

Degree/CN	Apex (m_0)	Rim (m_i)	Rim (l_i)
d_{G_v}	v	3	2
τ_{G_v}	v	$v-1$	3

Theorem 3.2. Let G_v be a gear graph with order $2v+1$. The generalized multiplicative ZI and the generalized first, second, third, and fourth multiplicative ZCIs of G_v are as follows:

1. $MZ_{\kappa,\lambda}(G_v) = [\mathrm{v}^{\kappa}3^{\lambda} + 3^{\kappa}v^{\lambda}]^{v} \times [3^{\kappa}2^{\lambda} + 2^{\kappa}3^{\lambda})^{2v}$,

2. $MC_{\kappa,\lambda}(G_v) = \left[\mathrm{v}^{\kappa}(v-1)^{\lambda} + (\mathrm{v}\text{-}1)^{\kappa}v^{\lambda}\right]^{v} \times \left[(v-1)^{\kappa}3^{\lambda} + 3^{\kappa}(v-1)^{\lambda}\right]^{2v}$,

3. $MC^{2}_{\kappa,\lambda}(G_v) = \left[\mathrm{v}^{\kappa}(v-1)^{\lambda} + 3^{\kappa}v^{\lambda}\right]^{\mathrm{r}} \times \left[3^{\kappa}3^{\lambda} + 2^{\kappa}(v-1)^{\lambda}\right]^{2v}$,

4. $MC^{3}_{\kappa,\lambda}(G_v) = \left[\mathrm{v}^{\kappa}v^{\lambda} + 3^{\kappa}(v-1)^{\lambda}\right]^{v} \times \left[3^{\kappa}(v-1)^{\lambda} + 2^{\kappa}3^{\lambda}\right]^{2v}$,

5. $MC^{4}_{\kappa,\lambda}(G_v) = 2^{2\kappa\mathrm{v}} \times 3^{(3\kappa+2\lambda)\mathrm{v}} \times r^{(\kappa+\lambda)\mathrm{v}} \times (r-1)^{3\lambda v}$.

Proof: 1. Since G_v is a gear graph of order $2v+1$. Also, m_0 is the apex vertex; $\{m_1, m_2, \ldots\ldots, m_v\}$ and $\{l_1, l_2, \ldots\ldots, l_v\}$ are the sets of rim vertices for G_v. Then, by definition,

$$MZ_{\kappa,\lambda}(G_v) = \prod_{lm\in E(G_v)} \left[d_{G_v}(l)^{\kappa} d_{G_v}(m)^{\lambda} + d_{G_v}(m)^{\kappa} d_{G_v}(l)^{\lambda}\right]$$

$$= \prod_{m_0 m_i\in E(G_v)} \left[d_{G_v}(m_0)^{\kappa} d_{G_v}(m_i)^{\lambda} + d_{G_v}(m_i)^{\kappa} d_{G_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i\in E(G_v)}$$
$$\left[d_{G_v}(m_i)^{\kappa} d_{G_v}(l_i)^{\lambda} + d_{G_v}(l_i)^{\kappa} d_{G_v}(m_i)^{\lambda}\right].$$

By using Table 2, we get

$$= [\mathrm{v}^{\kappa}3^{\lambda} + 3^{\kappa}v^{\lambda}]^{v} \times [3^{\kappa}2^{\lambda} + 2^{\kappa}3^{\lambda}]^{2v}.$$

2. $MC_{\kappa,\lambda}(G_v) = \prod_{lm\in E(G_v)} \left[\tau_{G_v}(l)^{\kappa} \tau_{G_v}(m)^{\lambda} + \tau_{G_v}(m)^{\kappa} \tau_{G_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(G_v)} \left[\tau_{G_v}(m_0)^{\kappa}\tau_{G_v}(m_i)^{\lambda} + \tau_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i \in E(G_v)} \left[\tau_{G_v}(m_i)^{\kappa}\tau_{G_v}(l_i)^{\lambda} + \tau_{G_v}(l_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda}\right]$$

$$= \left[\mathrm{v}^{\kappa}(v-1)^{\lambda} + (\mathrm{v}-1)^{\kappa}v^{\lambda}\right]^{r} \times \left[(v-1)^{\kappa}3^{\lambda} + 3^{\kappa}(v-1)^{\lambda}\right]^{2v}.$$

3. $MC^{2}_{\kappa,\lambda}(G_v) = \prod_{lm \in E(G_v)} \left[d_{G_v}(l)^{\kappa}\tau_{G_v}(m)^{\lambda} + d_{G_v}(m)^{\kappa}\tau_{G_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(G_v)} \left[d_{G_v}(m_0)^{\kappa}\tau_{G_v}(m_i)^{\lambda} + d_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i \in E(G_v)} \left[d_{G_v}(m_i)^{\kappa}\tau_{G_v}(l_i)^{\lambda} + d_{G_v}(l_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda}\right]$$

$$= \left[\mathrm{v}^{\kappa}(v-1)^{\lambda} + 3^{\kappa}v^{\lambda}\right]^{v} \times \left[3^{\kappa}3^{\lambda} + 2^{\kappa}(v-1)^{\lambda}\right]^{2v}.$$

4. $MC^{3}_{\kappa,\lambda}(G_v) = \prod_{lm \in E(G_v)} \left[d_{G_v}(l)^{\kappa}\tau_{G_v}(l)^{\lambda} + \tau_{G_v}(m)^{\kappa}d_{G_v}(m)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(G_v)} \left[d_{G_v}(m_0)^{\kappa}\tau_{G_v}(m_0)^{\lambda} + d_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda}\right] \times \prod_{m_i l_i \in E(G_v)} \left[d_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda} + d_{G_v}(l_i)^{\kappa}\tau_{G_v}(l_i)^{\lambda}\right]$$

$$= \left[\mathrm{v}^{\kappa}v^{\lambda} + 3^{\kappa}(v-1)^{\lambda}\right]^{v} \times \left[3^{\kappa}(v-1)^{\lambda} + 2^{\kappa}3^{\lambda}\right]^{2v}.$$

5. $MC^{4}_{\kappa,\lambda}(G_v) = \prod_{lm \in E(G_v)} \left[d_{G_v}(l)^{\kappa}\tau_{G_v}(l)^{\lambda} \times \tau_{G_v}(m)^{\kappa}d_{G_v}(m)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(G_v)} \left[d_{G_v}(m_0)^{\kappa}\tau_{G_v}(m_0)^{\lambda} \times d_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda}\right] \times \prod_{m_i l_i \in E(G_v)} \left[d_{G_v}(m_i)^{\kappa}\tau_{G_v}(m_i)^{\lambda} \times d_{G_v}(l_i)^{\kappa}\tau_{G_v}(l_i)^{\lambda}\right]$$

$$= [\mathrm{v}^{\kappa}v^{\lambda} \times 3^{\kappa}(v-1)^{\lambda}]^{v} \times [3^{\kappa}(v-1)^{\lambda} \times 2^{\kappa}3^{\lambda}]^{2v}$$

$$= 2^{2\alpha\mathrm{v}} \times 3^{(3\alpha+2\beta)\mathrm{v}} \times r^{(\alpha+\beta)\mathrm{v}} \times (r-1)^{3\lambda v}.$$

3.3. Helm Graph H_v

The helm graph H_v is obtained from the wheel graph by joining a pendant edge to every vertex of the rim. Thus, $2v+1$ and $3v$ are the order

$(|V(H_v)|)$ and size $(|E(H_v)|)$ of helm graph, respectively. Figure 3 depicts the graphical representation of helm graph.

Theorem 3.3. Let H_v be a helm graph with order $2v+1$. The generalized multiplicative ZI and the generalized first, second, third, and fourth multiplicative ZCIs of H_v are as follows:

1. $MZ_{\kappa,\lambda}(H_v) = 2^{\mathrm{v}}4^{(\kappa+\lambda)\mathrm{v}} \times (4^\kappa + 4^\lambda)^{\mathrm{v}} \times [\mathrm{r}^\kappa 4^\lambda + 4^\kappa r^\lambda]^v$,

2. $MC_{\kappa,\lambda}(H_v) = 2^{\mathrm{v}} \times (v-1)^{(\kappa+\lambda)\mathrm{v}} \times \left[v^\kappa (v-1)^\lambda + (\mathrm{v}\text{-}1)^\kappa v^\lambda\right]^r \times \left[(v-1)^\kappa 3^\lambda + 3^\kappa (v-1)^\lambda\right]^v$,

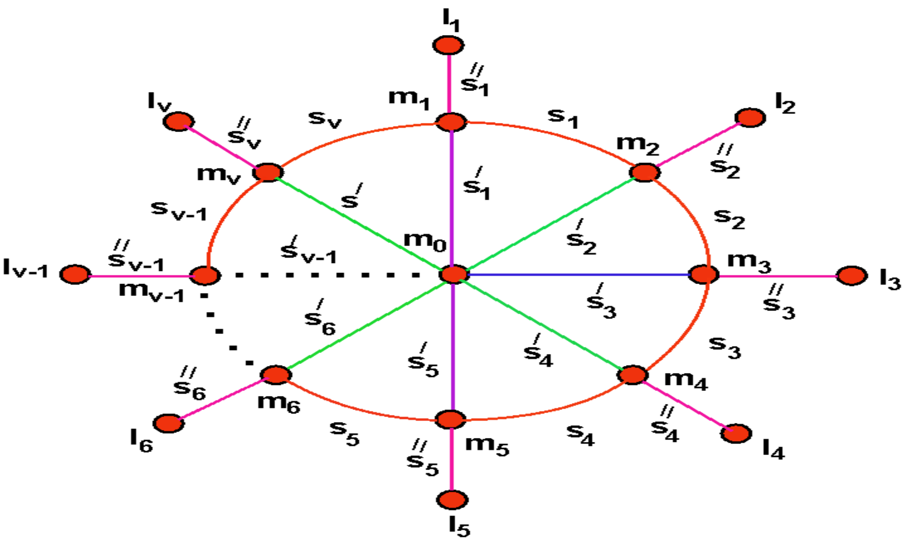

Figure 3. Helm Graph H_v

Table 3. Partition of Vertices for H_v

Degree/CN	Apex (m_0)	Rim (m_i)	Rim (l_i)
d_{H_v}	v	4	1
τ_{H_v}	v	$v-1$	3

3. $MC^2_{\kappa,\lambda}(H_v) = 2^{(1+2\kappa)\mathrm{v}} \times (v-1)^{\lambda v} \times \left[v^\kappa (v-1)^\lambda + 4^\kappa v^\lambda\right]^v \times \left[4^\kappa 3^\lambda + (\mathrm{v}\text{-}1)^\lambda\right]^v$,

4. $MC^3_{\kappa,\lambda}(H_v) = 2^{(1+2\kappa)\mathrm{v}} \times (v-1)^{\lambda v} \times \left[v^{\kappa+\lambda} + 4^\kappa (v-1)^\lambda\right]^v \times \left[4^\kappa (v-1)^\lambda + 3^\lambda\right]^v$,

5. $MC^4_{\kappa,\lambda}(H_v) = 3^{\lambda v} \times 4^{4\kappa v} \times r^{(\kappa+\lambda)v} \times (v-1)^{4\lambda v}$.

Proof: 1. Since H_v is a helm graph of order $2v+1$. Also, m_0 is the apex vertex; $\{m_1, m_2, \ldots\ldots, m_v\}$ and $\{l_1, l_2, \ldots\ldots, l_v\}$ are the sets of rim and pendant vertices for H_v, respectively. Then, by definition,

$$MZ_{\kappa,\lambda}(H_v) = \prod_{lm\in E(H_v)} \left[d_{H_v}(l)^\kappa d_{H_v}(m)^\lambda + d_{H_v}(m)^\kappa d_{H_v}(l)^\lambda\right]$$

$$= \prod_{m_0 m_i\in E(H_v)} \left[d_{H_v}(m_0)^\kappa d_{H_v}(m_i)^\lambda + d_{H_v}(m_i)^\kappa d_{H_v}(m_0)^\lambda\right] \times \prod_{m_i l_i\in E(H_v)} \left[d_{H_v}(m_i)^\kappa d_{H_v}(l_i)^\lambda + d_{H_v}(l_i)^\kappa d_{H_v}(m_i)^\lambda\right] \times \prod_{m_i m_{i+1}\in E(H_v)} \left[d_{H_v}(m_i)^\kappa d_{H_v}(m_{i+1})^\lambda + d_{H_v}(m_{i+1})^\kappa d_{H_v}(m_i)^\lambda\right].$$

By using Table 3, we get

$$= [v^\kappa 4^\lambda + 4^\kappa v^\lambda]^v \times [4^\kappa 1^\lambda + 1^\kappa 4^\lambda]^v \times [4^\kappa 4^\lambda + 4^\kappa 4^\lambda]^v$$

$$= 2^v 4^{(\kappa+\lambda)v} \times (4^\kappa + 4^\lambda)^v \times [v^\kappa 4^\lambda + 4^\kappa v^\lambda]^v.$$

2. $$MC_{\kappa,\lambda}(H_v) = \prod_{lm\in E(H_v)} \left[\tau_{H_v}(l)^\kappa \tau_{H_v}(m)^\lambda + \tau_{H_v}(m)^\kappa \tau_{H_v}(l)^\lambda\right]$$

$$= \prod_{m_0 m_i\in E(H_v)} \left[\tau_{H_r}(m_0)^\kappa \tau_{H_r}(m_i)^\lambda + \tau_{H_r}(m_i)^\kappa \tau_{H_r}(m_0)^\lambda\right] \times \prod_{m_i l_i\in E(H_r)} \left[\tau_{H_r}(m_i)^\kappa \tau_{H_r}(l_i)^\lambda + \tau_{H_r}(l_i)^\kappa \tau_{H_r}(m_i)^\lambda\right] \times \prod_{m_i m_{i+1}\in E(H_r)} \left[\tau_{H_v}(m_i)^\kappa \tau_{H_v}(m_{i+1})^\lambda + \tau_{H_v}(m_{i+1})^\kappa \tau_{H_v}(m_i)^\lambda\right]$$

$$= 2^v \times (v-1)^{(\kappa+\lambda)v} \times \left[v^\kappa (v-1)^\lambda + (v-1)^\kappa v^\lambda\right]^v \times \left[(v-1)^\kappa 3^\lambda + 3^\kappa (v-1)^\lambda\right]^v.$$

3. $$MC^2_{\kappa,\lambda}(H_v) = \prod_{lm\in E(H_v)} \left[d_{H_v}(l)^\kappa \tau_{H_v}(m)^\lambda + d_{H_v}(m)^\kappa \tau_{H_v}(l)^\lambda\right]$$

$$= \prod_{m_0 m_i\in E(H_v)} \left[d_{H_v}(m_0)^\kappa \tau_{H_v}(m_i)^\lambda + d_{H_v}(m_i)^\kappa \tau_{H_v}(m_0)^\lambda\right] \times \prod_{m_i l_i\in E(H_v)} \left[d_{H_v}(m_i)^\kappa \tau_{H_v}(l_i)^\lambda + d_{H_v}(l_i)^\kappa \tau_{H_v}(m_i)^\lambda\right] \times \prod_{m_i m_{i+1}\in E(H_v)}$$

$$\left[d_{H_v}(m_i)^{\kappa}\tau_{H_v}(m_{i+1})^{\lambda}+d_{H_v}(m_{i+1})^{\kappa}\tau_{H_v}(m_i)^{\lambda}\right]$$

$$=2^{(1+2\kappa)\mathrm{v}}\times(v-1)^{\lambda v}\times\left[v^{\kappa}(v-1)^{\lambda}+4^{\kappa}v^{\lambda}\right]^{v}\times\left[4^{\kappa}3^{\lambda}+(\mathrm{v}-1)^{\lambda}\right]^{v}.$$

4. $MC^{3}_{\kappa,\lambda}(H_v)=\prod_{lm\in E(H_v)}\left[d_{H_v}(l)^{\kappa}\tau_{H_v}(l)^{\lambda}+d_{H_v}(m)^{\kappa}\tau_{H_v}(m)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(H_v)}\left[d_{H_v}(m_0)^{\kappa}\tau_{H_v}(m_0)^{\lambda}+d_{H_v}(m_i)^{\kappa}\tau_{H_v}(m_i)^{\lambda}\right]\times\prod_{m_il_i\in E(H_r)}$$
$$\left[d_{H_r}(m_i)^{\kappa}\tau_{H_r}(m_i)^{\lambda}+d_{H_r}(l_i)^{\kappa}\tau_{H_r}(l_i)^{\lambda}\right]\times\prod_{m_im_{i+1}\in E(H_r)}$$
$$\left[d_{H_r}(m_i)^{\kappa}\tau_{H_r}(m_i)^{\lambda}+d_{H_r}(m_{i+1})^{\kappa}\tau_{H_r}(m_{i+1})^{\lambda}\right]$$

$$=2^{(1+2\kappa)\mathrm{r}}\times(r-1)^{\lambda r}\times\left[r^{\kappa+\lambda}+4^{\kappa}(r-1)^{\lambda}\right]^{r}\times\left[4^{\kappa}(r-1)^{\lambda}+3^{\lambda}\right]^{r}.$$

5. $MC^{4}_{\kappa,\lambda}(H_r)=\prod_{lm\in E(H_r)}\left[d_{H_r}(l)^{\kappa}\tau_{H_r}(l)^{\lambda}\times\tau_{H_r}(m)^{\kappa}d_{H_r}(m)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(H_r)}\left[d_{H_r}(m_0)^{\kappa}\tau_{H_r}(m_0)^{\lambda}\times d_{H_r}(m_i)^{\kappa}\tau_{H_r}(m_i)^{\lambda}\right]\times\prod_{m_il_i\in E(H_r)}$$
$$\left[d_{H_r}(m_i)^{\kappa}\tau_{H_r}(m_i)^{\lambda}\times d_{H_r}(l_i)^{\kappa}\tau_{H_r}(l_i)^{\lambda}\right]\times\prod_{m_im_{i+1}\in E(H_r)}$$
$$\left[d_{H_r}(m_i)^{\kappa}\tau_{H_r}(m_i)^{\lambda}\times d_{H_r}(m_{i+1})^{\kappa}\tau_{H_r}(m_{i+1})^{\lambda}\right]$$
$$=3^{\lambda\mathrm{r}}\times4^{4\kappa\mathrm{r}}\times r^{(\kappa+\lambda)\mathrm{r}}\times(r-1)^{4\lambda r}.$$

3.4. Flower Graph Fl_r

The flower graph Fl_r is obtained from the helm graph by joining each pendant vertex to the apex of helm graph. Thus, $2r+1$ and $4r$ are the order $(|V(Fl_r)|)$ and size $(|E(Fl_r)|)$ of flower graph, respectively. Figure 4 depicts the graphical representation of flower graph.

Table 4. Partition of Vertices for F_v

Degree/CN	Apex (m_0)	Rim (m_i)	Extreme (l_i)
d_{Fl_v}	$2v$	4	2
τ_{Fl_v}	0	$2v-4$	$2v-2$

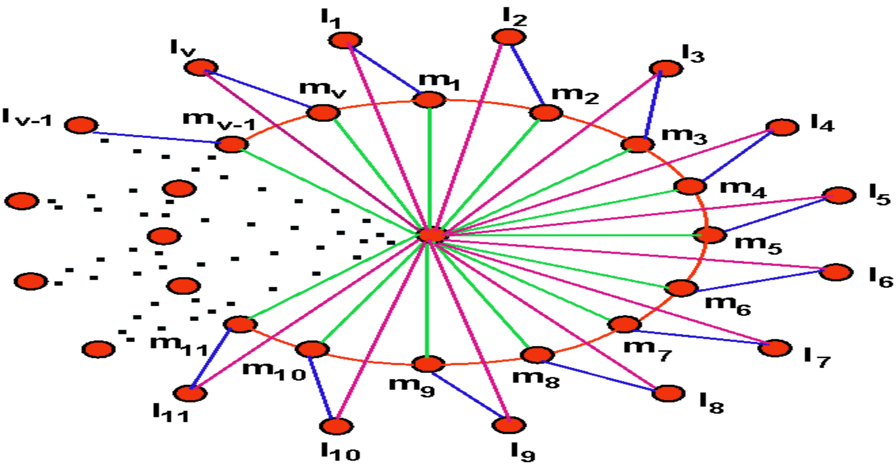

Figure 4. Flower Graph Fl_r

Theorem 3.4. Let Fl_v be a flower graph with order $2v+1$. The generalized multiplicative ZI and the generalized first, second, third, and fourth multiplicative ZCIs of Fl_v are as follows:

1. $MZ_{\kappa,\lambda}(Fl_v)=2^{(1+5\kappa+5\lambda)\mathrm{v}}\times(2^{\kappa}+2^{\lambda})^{\mathrm{v}}\times[\mathrm{r}^{\kappa}2^{\lambda}+2^{\kappa}r^{\lambda}]^{v}$,

2. $MC_{\kappa,\lambda}(Fl_v)=2^{(1+2\kappa+2\lambda)\mathrm{v}}\times(v-2)^{(\kappa+\lambda)\mathrm{v}}\times\left[(v-2)^{\kappa}(\mathrm{v}-1)^{\lambda}+(\mathrm{v}-1)^{\kappa}(v-2)^{\lambda}\right]^{v}$,

3. $MC^{2}_{\kappa,\lambda}(Fl_v)=2^{(1+5\kappa+4\lambda)\mathrm{v}}\times\left[v^{2\kappa}(v-1)^{\lambda}(\mathrm{v}-2)^{2\lambda}\right]^{r}\times\left[2^{\kappa}(v-1)^{\lambda}+(\mathrm{v}-2)^{\lambda}\right]^{v}$,

4. $MC^{3}_{\kappa,\lambda}(Fl_v)=2^{(1+6\kappa+4\lambda)\mathrm{v}}\times(r-1)^{\lambda v}\times(r-2)^{2\lambda v}\times\left[2^{\kappa}(v-2)^{\lambda}+(\mathrm{v}-1)^{\lambda}\right]^{v}$,

5. $MC^{4}_{\kappa,\lambda}(Fl_v)=0$.

Proof: 1. Since Fl_v is a flower graph of order $2v+1$. Also, m_0 is the apex vertex; $\{m_1,m_2,\ldots\ldots,m_v\}$ and $\{l_1,l_2,\ldots\ldots,l_v\}$ are the sets of rim and extreme vertices for Fl_v, respectively. Then, by definition,

$$MZ_{\kappa,\lambda}(Fl_v)=\prod_{lm\in E(Fl_v)}\left[d_{H_v}(l)^{\kappa}d_{Fl_v}(m)^{\lambda}+d_{Fl_v}(m)^{\kappa}d_{Fl_v}(l)^{\lambda}\right]$$

$$=\prod_{m_0m_i\in E(Fl_v)}\left[d_{Fl_v}(m_0)^{\kappa}d_{Fl_v}(m_i)^{\lambda}+d_{Fl_v}(m_i)^{\kappa}d_{Fl_v}(m_0)^{\lambda}\right]\times\prod_{m_0l_i\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_0)^{\kappa} d_{Fl_v}(l_i)^{\lambda} + d_{Fl_v}(l_i)^{\kappa} d_{Fl_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i \in E(Fl_v)} \left[d_{Fl_v}(m_i)^{\kappa} d_{Fl_v}(l_i)^{\lambda} + d_{Fl_v}(l_i)^{\kappa} d_{Fl_v}(m_i)^{\lambda}\right]$$

$$\times \prod_{m_i m_{i+1} \in E(Fl_v)} \left[d_{Fl_v}(m_i)^{\kappa} d_{Fl_v}(m_{i+1})^{\lambda} + d_{Fl_v}(m_{i+1})^{\kappa} d_{Fl_v}(m_i)^{\lambda}\right].$$

By using Table 4, we get

$$=[(2\mathrm{v})^{\kappa} 4^{\lambda} + 4^{\kappa}(2v)^{\lambda}]^{\mathrm{v}} \times [(2\mathrm{v})^{\kappa} 2^{\lambda} + 2^{\kappa}(2v)^{\lambda}]^{v} \times [4^{\kappa} 2^{\lambda} + 2^{\kappa} 4^{\lambda}]^{v} \times [4^{\kappa} 4^{\lambda} + 4^{\kappa} 4^{\lambda}]^{\mathrm{v}} = 2^{(1+5\kappa+5\lambda)\mathrm{v}} \times (2^{\kappa} + 2^{\lambda})^{\mathrm{v}} \times [\mathrm{v}^{\kappa} 2^{\lambda} + 2^{\kappa} v^{\lambda}]^{\mathrm{v}}.$$

2. $MC_{\kappa,\lambda}(Fl_v) = \prod_{lm \in E(Fl_v)} \left[\tau_{Fl_v}(l)^{\kappa} \tau_{Fl_v}(m)^{\lambda} + \tau_{Fl_v}(m)^{\kappa} \tau_{Fl_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(Fl_v)} \left[\tau_{Fl_v}(m_0)^{\kappa} \tau_{Fl_v}(m_i)^{\lambda} + \tau_{Fl_v}(m_i)^{\kappa} \tau_{Fl_v}(m_0)^{\lambda}\right] \times \prod_{m_0 l_i \in E(Fl_v)} \left[\tau_{Fl_v}(m_0)^{\kappa} \tau_{Fl_v}(l_i)^{\lambda} + \tau_{Fl_v}(l_i)^{\kappa} \tau_{Fl_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i \in E(Fl_v)} \left[\tau_{Fl_v}(m_i)^{\kappa} \tau_{Fl_v}(l_i)^{\lambda} + \tau_{Fl_v}(l_i)^{\kappa} \tau_{Fl_v}(m_i)^{\lambda}\right]$$

$$\times \prod_{m_i m_{i+1} \in E(Fl_v)} \left[\tau_{Fl_v}(m_i)^{\kappa} \tau_{Fl_v}(m_{i+1})^{\lambda} + \tau_{Fl_v}(m_{i+1})^{\kappa} \tau_{Fl_v}(m_i)^{\lambda}\right]$$

$$= \left[0^{\kappa}(2v-4)^{\lambda} + (2\mathrm{v}-4)^{\kappa} 0^{\lambda}\right]^{v} \times \left[0^{\kappa}(2v-2)^{\lambda} + (2\mathrm{v}-2)^{\kappa} 0^{\lambda}\right]^{v} \times \left[(2v-4)^{\kappa}(2v-2)^{\lambda} + (2\mathrm{v}-2)^{\kappa}(2\mathrm{v}-4)^{\lambda}\right]^{v} \times \left[(2v-4)^{\kappa}(2v-4)^{\lambda} + (2\mathrm{v}-4)^{\kappa}(2\mathrm{v}-4)^{\lambda}\right]^{v}$$

$$= 2^{(1+2\kappa+2\lambda)\mathrm{v}} \times (v-2)^{(\kappa+\lambda)\mathrm{v}} \times \left[(v-2)^{\kappa}(v-1)^{\lambda} + (\mathrm{v}-1)^{\kappa}(\mathrm{v}-2)^{\lambda}\right]^{v}$$

3. $MC^{2}_{\kappa,\lambda}(Fl_v) = \prod_{lm \in E(Fl_v)} \left[d_{Fl_v}(l)^{\kappa} \tau_{Fl_v}(m)^{\lambda} + d_{Fl_v}(m)^{\kappa} \tau_{Fl_v}(l)^{\lambda}\right]$

$$= \prod_{m_0 m_i \in E(Fl_v)} \left[d_{Fl_v}(m_0)^{\kappa} \tau_{Fl_v}(m_i)^{\lambda} + d_{Fl_v}(m_i)^{\kappa} \tau_{Fl_v}(m_0)^{\lambda}\right] \times \prod_{m_0 l_i \in E(Fl_v)} \left[d_{Fl_v}(m_0)^{\kappa} \tau_{Fl_v}(l_i)^{\lambda} + d_{Fl_v}(l_i)^{\kappa} \tau_{Fl_v}(m_0)^{\lambda}\right] \times \prod_{m_i l_i \in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(l_i)^{\lambda}+d_{Fl_v}(l_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}\right]\times\prod_{m_im_{i+1}\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_{i+1})^{\lambda}+d_{Fl_v}(m_{i+1})^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}\right]$$

$$=\left[(2v)^{\kappa}(2v-4)^{\lambda}+4^{\kappa}0^{\lambda}\right]^{v}\times\left[(2v)^{\kappa}(2v-2)^{\lambda}+2^{\kappa}0^{\lambda}\right]^{v}\times$$

$$\left[4^{\kappa}(2v-2)^{\lambda}+2^{\kappa}(2v-4)^{\lambda}\right]^{v}\times\left[4^{\kappa}(2v-4)^{\lambda}+4^{\kappa}(2v-4)^{\lambda}\right]^{v}$$

$$=2^{(1+5\kappa+4\lambda)v}\times\left[r^{2\kappa}(v-1)^{\lambda}(v-2)^{2\lambda}\right]^{v}\times\left[2^{\kappa}(v-1)^{\lambda}+(v-2)^{\lambda}\right]^{v}.$$

4. $MC^{3}_{\kappa,\lambda}(Fl_v)=\prod_{lm\in E(Fl_v)}\left[d_{Fl_v}(l)^{\kappa}\tau_{Fl_v}(l)^{\lambda}+d_{Fl_v}(m)^{\kappa}\tau_{Fl_v}(m)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(Fl_v)}\left[d_{Fl_v}(m_0)^{\kappa}\tau_{Fl_v}(m_0)^{\lambda}+d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}\right]\times\prod_{m_0l_i\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_0)^{\kappa}\tau_{Fl_v}(m_0)^{\lambda}+d_{Fl_v}(l_i)^{\kappa}\tau_{Fl_v}(l_i)^{\lambda}\right]\times\prod_{m_il_i\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}+d_{Fl_v}(l_i)^{\kappa}\tau_{Fl_v}(l_i)^{\lambda}\right]\times\prod_{m_im_{i+1}\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}+d_{Fl_v}(m_{i+1})^{\kappa}\tau_{Fl_v}(m_{i+1})^{\lambda}\right]$$

$$=\left[(2v)^{\kappa}0^{\lambda}+4^{\kappa}(2v-4)^{\lambda}\right]^{v}\times\left[(2v)^{\kappa}0^{\lambda}+2^{\kappa}(2v-2)^{\lambda}\right]^{v}\times$$

$$\left[4^{\kappa}(2v-4)^{\lambda}+2^{\kappa}(2v-2)^{\lambda}\right]^{v}\times\left[4^{\kappa}(2v-4)^{\lambda}+4^{\kappa}(2v-4)^{\lambda}\right]^{v}$$

$$=2^{(1+6\kappa+4\lambda)v}\times(v-1)^{\lambda v}\times(v-2)^{2\lambda v}\times\left[2^{\kappa}(v-2)^{\lambda}+(v-1)^{\lambda}\right]^{v}.$$

5. $MC^{4}_{\kappa,\lambda}(Fl_v)=\prod_{lm\in E(Fl_v)}\left[d_{Fl_v}(l)^{\kappa}\tau_{Fl_v}(l)^{\lambda}\times\tau_{Fl_v}(m)^{\kappa}d_{Fl_v}(m)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(Fl_v)}\left[d_{Fl_v}(m_0)^{\kappa}\tau_{Fl_v}(m_0)^{\lambda}\times d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}\right]\times\prod_{m_0l_i\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_0)^{\alpha}\tau_{Fl_v}(m_0)^{\beta}\times d_{Fl_v}(l_i)^{\alpha}\tau_{Fl_v}(l_i)^{\beta}\right]\times\prod_{m_il_i\in E(Fl_v)}$$

$$\left[d_{Fl_v}(m_i)^{\alpha}\tau_{Fl_v}(m_i)^{\beta}\times d_{Fl_v}(l_i)^{\alpha}\tau_{Fl_v}(l_i)^{\beta}\right]$$

$$\times\prod_{m_im_{i+1}\in E(Fl_v)}\left[d_{Fl_v}(m_i)^{\kappa}\tau_{Fl_v}(m_i)^{\lambda}\times d_{Fl_v}(m_{i+1})^{\kappa}\tau_{Fl_v}(m_{i+1})^{\lambda}\right]$$

$$= \left[(2v)^{\kappa}0^{\lambda} \times 4^{\kappa}(2v-4)^{\lambda}\right]^{v} \times \left[(2v)^{\kappa}0^{\lambda} \times 2^{\kappa}(2v-2)^{\lambda}\right]^{v} \times$$

$$\left[4^{\kappa}(2v-4)^{\lambda} \times 2^{\kappa}(2v-2)^{\lambda}\right]^{v} \times \left[4^{\kappa}(2v-4)^{\lambda} \times 4^{\kappa}(2v-4)^{\lambda}\right]^{v}$$

$$= 0.$$

3.5. Sunflower Graph Sf_v

The sunflower graph Sf_v is obtained from the flower graph by growing v pendant edges to the apex of flower graph. Thus, $3v+1$ and $5v$ are the order $(|V(Sf_v)|)$ and size $(|E(Sf_v)|)$ of sunflower graph, respectively. Figure 5 depicts the graphical representation of sunflower graph.

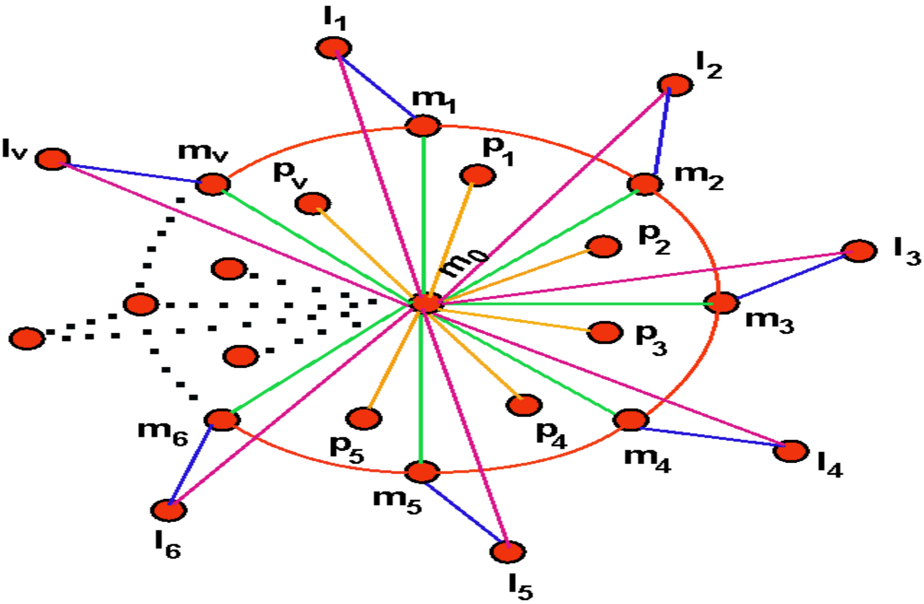

Figure 5. Sunflower Graph Sf_v

Table 5. Partition of Vertices for Sf_v

Degree/CN	Apex (m_0)	Rim (m_i)	Extreme (l_i)	Pendant (p_i)
d_{Sf_v}	$3v$	4	2	1
τ_{Sfv}	0	$3v-4$	$3v-2$	$3v-1$

Theorem 3.5. Let Sf_v be a flower graph with order $3v+1$. The generalized multiplicative ZI and the generalized first, second, third, and fourth multiplicative ZCIs of Sf_v are as follows:

1. $MZ_{\kappa,\lambda}(Sf_v) = 2^{(1+3\kappa+3\lambda)\text{v}} \times (2^{\kappa} + 2^{\lambda})^{\text{v}} \times [(3\text{v})^{\kappa} + (3\text{v})^{\lambda}]^{v} \times \left[(3v)^{\kappa} 4^{\lambda} + 4^{\kappa}(3v)^{\lambda}\right]^{v}$

$\times [(3v)^{\kappa} 2^{\lambda} + 2^{\kappa}(3v)^{\lambda}]^{v}$,

2. $MC_{\kappa,\lambda}(Sf_v) = 2^{\text{v}} \times (3v-4)^{(\kappa+\lambda)\text{v}} \times \left[(3v-4)^{\kappa}(3\text{v}-2)^{\lambda} + (3\text{v}-2)^{\kappa}(3v-4)^{\lambda}\right]^{v}$ **,**

3. $MC^{2}_{\kappa,\lambda}(Sf_v) = 2^{\text{r}} \times (6r)^{3\kappa v} \times (3r-4)^{\lambda v} \times \left[(3v-1)(3\text{v}-2)(3\text{v}-4)\right]^{\lambda v} \times \left[2^{\kappa}(3v-2)^{\lambda} + (3\text{v}-4)^{\lambda}\right]^{v}$ **,**

4. $MC^{3}_{\kappa,\lambda}(Sf_v) = 2^{(1+6\kappa)\text{v}} \times \left[(3v-1)(3v-2)(3v-4)^{2}\right]^{\lambda v} \times \left[2^{\kappa}(3v-4)^{\lambda} + (3\text{v}-2)^{\lambda}\right]^{v}$**,**

5. $MC^{4}_{\kappa,\lambda}(Sf_v) = 0$**.**

Proof: 1 Since Sf_v is a sunflower graph of order $3v+1$. Also, m_0 is the apex vertex; $\{m_1, m_2, \ldots\ldots, m_v\}$, $\{l_1, l_2, \ldots\ldots, l_v\}$, and $\{p_1, p_2, \ldots\ldots, p_v\}$ are the sets of rim and extreme and pendant vertices for Sf_v, respectively. Then, by definition,

$$MZ_{\kappa,\lambda}(Sf_v) = \prod_{lm\in E(Sf_v)} \left[d_{Sf_v}(l)^{\kappa} d_{Sf_v}(m)^{\lambda} + d_{Sf_v}(m)^{\kappa} d_{Sf_v}(l)^{\lambda}\right]$$

$$= \prod_{m_0 m_i\in E(Sf_v)} \left[d_{Sf_v}(m_0)^{\kappa} d_{Sf_v}(m_i)^{\lambda} + d_{Sf_v}(m_i)^{\kappa} d_{Sf_v}(m_0)^{\lambda}\right] \times \prod_{m_0 l_i\in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa} d_{Sf_v}(l_i)^{\lambda} + d_{Sf_v}(l_i)^{\kappa} d_{Sf_v}(m_0)^{\lambda}\right] \times \prod_{m_0 p_i\in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa} d_{Sf_v}(p_i)^{\lambda} + d_{Sf_v}(p_i)^{\kappa} d_{Sf_v}(m_0)^{\lambda}\right]$$

$$\times \prod_{m_i l_i\in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\kappa} d_{Sf_v}(l_i)^{\lambda} + d_{Sf_v}(l_i)^{\kappa} d_{Sf_v}(m_i)^{\lambda}\right] \times$$

$$\prod_{m_i m_{i+1}\in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\kappa} d_{Sf_v}(m_{i+1})^{\lambda} + d_{Sf_v}(m_{i+1})^{\kappa} d_{Sf_v}(m_i)^{\lambda}\right].$$

By using Table 5, we get

$$= [(3\text{v})^{\kappa} 4^{\lambda} + 4^{\kappa}(3v)^{\lambda}]^{v} \times [(3\text{v})^{\kappa} 2^{\lambda} + 2^{\kappa}(3v)^{\lambda}]^{v} \times$$

$$[(3\mathrm{v})^{\kappa}1^{\lambda}+1^{\kappa}(3v)^{\lambda}]^{\mathrm{v}}\times[4^{\kappa}2^{\lambda}+2^{\kappa}4^{\lambda}]^{\mathrm{v}}$$

$$\times[4^{\kappa}4^{\lambda}+4^{\kappa}4^{\lambda}]^{v}$$

$$=2^{(1+3\kappa+3\lambda)\mathrm{v}}\times(2^{\kappa}+2^{\lambda})^{\mathrm{v}}\times[(3\mathrm{v})^{\kappa}+(3\mathrm{v})^{\lambda}]^{v}\times\left[(3v)^{\kappa}4^{\lambda}+4^{\kappa}(3v)^{\lambda}\right]^{v}$$
$$\times[(3v)^{\kappa}2^{\lambda}+2^{\kappa}(3v)^{\lambda}]^{v}.$$

2. $MC_{\kappa,\lambda}(Sf_v)=\prod_{lm\in E(Sf_v)}\left[\tau_{Sf_v}(l)^{\kappa}\tau_{Sf_v}(m)^{\lambda}+\tau_{Sf_v}(m)^{\kappa}\tau_{Sf_v}(l)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(Sf_v)}\left[\tau_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}+\tau_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]\times\prod_{m_0l_i\in E(Sf_v)}$$

$$\left[\tau_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(l_i)^{\lambda}+\tau_{Sf_v}(l_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]\times\prod_{m_0p_i\in E(Sf_v)}$$

$$\left[\tau_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(p_i)^{\lambda}+\tau_{Sf_v}(p_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]$$

$$\times\prod_{m_il_i\in E(Sf_v)}\left[\tau_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(l_i)^{\lambda}+\tau_{Sf_v}(l_i)^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}\right]\times$$

$$\prod_{m_im_{i+1}\in E(Sf_v)}\left[\tau_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(m_{i+1})^{\lambda}+\tau_{Sf_v}(m_{i+1})^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}\right]$$

$$=[0^{\kappa}(3v-4)^{\lambda}+(3\mathrm{v}-4)^{\kappa}0^{\lambda}]^{v}\times\ [0^{\kappa}(3v-2)^{\lambda}+(3\mathrm{v}-2)^{\kappa}0^{\lambda}]^{v}\ \times$$
$$[0^{\kappa}(3v-1)^{\lambda}+(3\mathrm{v}-1)^{\kappa}0^{\lambda}]^{r}\times[(3v-4)^{\kappa}(3v-2)^{\lambda}+(3v-2)^{\kappa}(3v-4)^{\lambda}]^{\mathrm{v}}$$

$$\times\left[(3v-4)^{\kappa}(3v-2)^{\lambda}+(3\mathrm{v}-2)^{\kappa}(3v-4)^{\lambda}\right]^{v}$$

$$=2^{\mathrm{v}}\times(3v-4)^{(\kappa+\lambda)\mathrm{v}}\times\left[(3v-4)^{\kappa}(3\mathrm{v}-2)^{\lambda}+(3\mathrm{v}-2)^{\kappa}(3v-4)^{\lambda}\right]^{v}.$$

3. $MC^{2}_{\kappa,\lambda}(Sf_v)=\prod_{lm\in E(Sf_v)}\left[d_{Sf_v}(l)^{\kappa}\tau_{Sf_v}(m)^{\lambda}+d_{Sf_v}(m)^{\kappa}\tau_{Sf_v}(l)^{\lambda}\right]$

$$=\prod_{m_0m_i\in E(Sf_v)}\left[d_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}+d_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]\times\prod_{m_0l_i\in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(l_i)^{\lambda}+d_{Sf_v}(l_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]\times\prod_{m_0p_i\in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(p_i)^{\lambda}+d_{Sf_v}(p_i)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\right]$$

$$\times \prod_{m_i l_i \in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\kappa} \tau_{Sf_v}(l_i)^{\lambda} + d_{Sf_v}(l_i)^{\kappa} \tau_{Sf_v}(m_i)^{\lambda} \right] \times$$

$$\prod_{m_i m_{i+1} \in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\alpha} \tau_{Sf_v}(m_{i+1})^{\beta} + d_{Sf_v}(m_{i+1})^{\alpha} \tau_{Sf_v}(m_i)^{\beta} \right]$$

$$= [(3\text{v})^{\alpha}(3v-4)^{\beta} + 4^{\alpha}0^{\beta}]^{\text{v}} \times [(3\text{v})^{\kappa}(3v-2)^{\lambda} + 2^{\kappa}0^{\lambda}]^{\text{v}} \times$$
$$[(3\text{v})^{\kappa}(3v-1)^{\lambda} + 1^{\kappa}0^{\lambda}]^{\text{v}} \times [4^{\kappa}(3v-2)^{\lambda} + 2^{\kappa}(3v-4)^{\lambda}]^{\text{v}}$$

$$\times \left[4^{\kappa}(3v-4)^{\lambda} + 4^{\kappa}(3v-4)^{\lambda}\right]^{v}$$

$$= 2^{\text{v}} \times (6r)^{3\kappa v} \times (3v-4)^{\lambda v} \times \left[(3v-1)(3\text{v-}2)(3\text{v-}4)\right]^{\lambda v} \times \left[2^{\kappa}(3v-2)^{\lambda} + (3\text{v-}4)^{\lambda}\right]^{v}$$

4. $MC^{3}_{\kappa,\lambda}(Sf_v) = \prod_{lm \in E(Sf_v)} \left[d_{Sf_v}(l)^{\kappa} \tau_{Sf_v}(l)^{\lambda} + d_{Sf_v}(m)^{\kappa} \tau_{Sf_v}(m)^{\lambda} \right]$

$$= \prod_{m_0 m_i \in E(Sf_v)} \left[d_{Sf_v}(m_0)^{\kappa} \tau_{Sf_v}(m_0)^{\lambda} + d_{Sf_v}(m_i)^{\kappa} \tau_{Sf_v}(m_i)^{\lambda} \right] \times \prod_{m_0 l_i \in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa} \tau_{Sf_v}(m_0)^{\lambda} + d_{Sf_v}(l_i)^{\kappa} \tau_{Sf_v}(l_i)^{\lambda} \right] \times \prod_{m_0 p_i \in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa} \tau_{Sf_v}(m_0)^{\lambda} + d_{Sf_v}(p_i)^{\kappa} \tau_{Sf_v}(p_i)^{\lambda} \right]$$

$$\times \prod_{m_i l_i \in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\kappa} \tau_{Sf_v}(m_i)^{\lambda} + d_{Sf_v}(l_i)^{\kappa} \tau_{Sf_v}(l_i)^{\lambda} \right] \times$$

$$\prod_{m_i m_{i+1} \in E(Sf_v)} \left[d_{Sf_v}(m_i)^{\kappa} \tau_{Sf_v}(m_i)^{\lambda} + d_{Sf_v}(m_{i+1})^{\kappa} \tau_{Sf_v}(m_{i+1})^{\lambda} \right]$$

$$= [(3\text{v})^{\kappa}0^{\lambda} + 4^{\kappa}(3v-4)^{\lambda}]^{\text{v}} \times [(3\text{v})^{\kappa}0^{\lambda} + 2^{\kappa}(3v-2)^{\lambda}]^{\text{v}} \times$$
$$[(3\text{v})^{\kappa}0^{\lambda} + 1^{\kappa}(3v-1)^{\lambda}]^{r} \times [4^{\kappa}(3v-4)^{\lambda} + 2^{\kappa}(3v-2)^{\lambda}]^{\text{v}}$$

$$\times \left[4^{\kappa}(3v-4)^{\lambda} + 4^{\kappa}(3v-4)^{\lambda}\right]^{v}$$

$$= 2^{(1+6\kappa)\text{v}} \times \left[(3v-1)(3v-2)(3v-4)^{2}\right]^{\lambda v} \times \left[2^{\kappa}(3v-4)^{\lambda} + (3\text{v-}2)^{\lambda}\right]^{v}.$$

5. $MC^{4}_{\kappa,\lambda}(Sf_v) = \prod_{lm \in E(Sf_r)} \left[d_{Sf_v}(l)^{\kappa} \tau_{Sf_v}(l)^{\lambda} \times d_{Sf_v}(m)^{\kappa} \tau_{Sf_v}(m)^{\lambda} \right]$

$$= \prod_{m_0 m_i \in E(Sf_v)} \left[d_{Sf_v}(m_0)^{\kappa} \tau_{Sf_v}(m_0)^{\lambda} \times d_{Sf_v}(m_i)^{\kappa} \tau_{Sf_v}(m_i)^{\lambda} \right] \times \prod_{m_0 l_i \in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\times d_{Sf_v}(l_i)^{\kappa}\tau_{Sf_v}(l_i)^{\lambda}\right]\times\prod_{m_0p_i\in E(Sf_v)}$$

$$\left[d_{Sf_v}(m_0)^{\kappa}\tau_{Sf_v}(m_0)^{\lambda}\times d_{Sf_v}(p_i)^{\kappa}\tau_{Sf_v}(p_i)^{\lambda}\right]$$

$$\times\prod_{m_il_i\in E(Sf_v)}\left[d_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}\times d_{Sf_v}(l_i)^{\kappa}\tau_{Sf_v}(l_i)^{\lambda}\right]\times$$

$$\prod_{m_im_{i+1}\in E(Sf_v)}\left[d_{Sf_v}(m_i)^{\kappa}\tau_{Sf_v}(m_i)^{\lambda}\times d_{Sf_v}(m_{i+1})^{\kappa}\tau_{Sf_v}(m_{i+1})^{\lambda}\right]$$

$$=[(3\mathrm{v})^{\kappa}0^{\lambda}\times4^{\kappa}(3v-4)^{\lambda}]^{v}\times[(3\mathrm{v})^{\kappa}0^{\lambda}\times2^{\kappa}(3v-2)^{\lambda}]^{v}\times$$

$$[(3\mathrm{v})^{\kappa}0^{\lambda}\times1^{\kappa}(3v-1)^{\lambda}]^{r}\times[4^{\kappa}(3v-4)^{\lambda}\times2^{\kappa}(3v-2)^{\lambda}]^{\mathrm{v}}$$

$$\times\left[4^{\kappa}(3v-4)^{\lambda}\times4^{\kappa}(3v-4)^{\lambda}\right]^{v}$$

$$=0.$$

4. CONCLUSION

Topological indices (TIs) are the mathematical coding of molecular graphs that predict the physicochemical, toxicological, biological, and structural properties of chemical compounds that are directly linked with these graphs. The Zagreb connection indices (ZCIs) are among the TIs of molecular graphs that depend upon the connection number (CN). These CN-based TIs are well used in the study of quantitative structures activity relationships (QSARs) and quantitative structures property relationships (QSPRs). These days, CN-based multiplicative Zagreb indices are the best tools available for the study of QSARs and QSPRs. In this paper, the generalized first, second, third, and fourth multiplicative ZCIs were defined. Also, some exact solutions of the novel/old generalized multiplicative Zagreb indices for some special graphs, namely flower, sunflower, wheel, helm, and gear graphs were computed.

CONFLICT OF INTEREST

The authors of the manuscript have no financial or non-financial conflict of interest in the subject matter or materials discussed in this manuscript.

DATA AVALIABILITY STATEMENT

Data availability is not applicable as no new data was created.

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