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A Study of New Continuous Univariate Weibull Model with Empirical Illustrations

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ABSTRACT

The theory of probability distribution is used for mathematical modeling in various research areas. It makes those models flexible that are used to quantify uncertainties and risks in extreme events. This is also important for decision-making. This study proposes a new model labeled as 'exponentiated alpha Weibull (EAW)'. The densities and reliability functions of EAW are also depicted graphically. Various statistical properties of the proposed model are explored. For estimation, the maximum likelihood (ML) method is applied. Simulation study is also performed. To check model performance, the proposed model is compared with other competitive models (CMs) for water runoff data.

Keywords: alpha, empirical, exponentiated, extreme, hazard, likelihood, runoff, simulation, Weibull

1. INTRODUCTION

The theory of statistical probability distribution of extreme hydrological events has a fundamental role in designing infrastructure, risk assessment, response in emergency, climate adaptation, and the protection of environment. It provides a structured and quantitative approach to understand the occurrence and impact of extreme events, enabling informed decision-making, and enhancing societal resilience. In reliability theory, the Weibull model, presented by [1], is a very famous lifetime model. It has an advantage over the exponential model due to its decreasing and increasing trends of hazard rate function (HRFn) depending on shape parameter. Usually, it is employed for analyzing biological, reliability and survival, hydrological, medical, and failure time of equipment datasets. It provides an inappropriate fit for several datasets, particularly when the

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shapes of HRFn are upside down bathtub, bathtub, or bimodal. To overcome these situations, several researchers have attempted various generalizations and extensions of the Weibull distribution (WDn) to improve model performance by injecting more parameters. Among these, the most useful distributions are the exponentiated Weibull (EW) that allows bathtub-shaped HRFn by [2, 3], additive Weibull by [4], exponentiated Weibull by [5] with flood applications, modified Weibull (MW) by [6], extended Weibull by [7], flexible MW by [8], generalized MW by [9], exponential-Weibull by [10], Weibull-G family by [11], beta Sarhan-Zaindin MW by [12], and exponentiated Weibull by [13]. For readers, we suggest the latest extensions by [14–20].

[21] developed a transformation known as alpha power (APr) to manipulate non-symmetrical patterns of continuous classical distributions. When a random variable (rv) X follows an APr transformation, then its cumulative distribution function (CDFn) and probability density function (PDFn) are as follows:

$$\Phi_{APr}(x;\alpha,\Xi)=\begin{cases}\dfrac{\alpha^{F(x;\Xi)}-1}{\alpha-1}: & x>0,\alpha>0,\alpha\neq 1,\\ F(x;\Xi): & \alpha=1.\end{cases}$$

And

$$\phi_{APr}(x;\alpha,\Xi)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\mathrm{f}(x;\Xi)\alpha^{F(x;\Xi)}: & x>0,\alpha>0,\alpha\neq 1,\\ \mathrm{f}(x;\Xi): & \alpha=1.\end{cases}$$

An extra parameter in the form of the index introduced by [22] through exponentiated family (EFy) has CDFn and PDFn as follows:

$$\Phi_{EFy}(x;\lambda,\Xi)=F^{\lambda}(x;\Xi),$$

and

$$\phi_{EFy}(x;\lambda,\Xi)=\lambda f(x;\Xi)F^{\lambda-1}(x;\Xi): x>0,\lambda>0.$$

For the generalization of the parent distributions, [23] developed T-X family of distributions with CDFn and PDFn as follows:

$$\Phi_{TXF}(x;\Xi)=\int_{\sigma_1}^{\mathrm{H}[F(x;\Xi)]}\pi(t)dt=\Pi\{\mathrm{H}[F(x;\Xi)]\},$$

and

$$\phi_{TXF}(x;\Xi)=\pi\{\mathrm{H}[F(x;\ \Xi)]\}\frac{\partial}{\partial x}\mathrm{H}[F(x;\ \Xi)].$$

$\pi(t)$ is the PDFn of random variable $\mathrm{T}\in[\sigma_1,\sigma_2]$, such that $-\infty<\sigma_1<\sigma_2<+\infty$. While, $F(x;\ \Xi)$ is the CDFn of random variable X with link function $\mathrm{H}(.):[0,1]\rightarrow[\sigma_1,\sigma_2]$ having the following conditions:

- $\mathrm{H}(.)$ has a monotonically increasing and differentiable function.
- $\mathrm{H}(.)\rightarrow\begin{cases}\sigma_1: & \lim\limits_{x\to-\infty}F(x;\Xi),\\ \sigma_2: & \lim\limits_{x\to+\infty}F(x;\Xi).\end{cases}$

[24] developed exponentiated power alpha index (EPAIx) generalized family with CDFn and PDFn as follows:

$$\Phi_{EPAIx}(x;\alpha,\theta,\Xi)=\left[\frac{\alpha^{\frac{-\log\{1-F(x;\Xi)\}}{1-\log\{1-F(x;\Xi)\}}}-1}{\alpha-1}\right]^{\theta};\ \alpha,\theta,x>0,\alpha\neq 1, \tag{1.1}$$

and

$$\phi_{EPAIx}(x;\ \alpha,\theta,\Xi)$$
$$=\frac{\theta\log\alpha}{(\alpha-1)}\frac{f(x;\ \Xi)}{\{1-F(x;\ \Xi)\}[1-\log\{1-F(x;\ \Xi)\}]^2}\left[\frac{\alpha^{\frac{-\log\{1-F(x;\Xi)\}}{1-\log\{1-F(x;\Xi)\}}}-1}{\alpha-1}\right]^{\theta-1}\alpha^{\frac{-\log\{1-F(x;\Xi)\}}{1-\log\{1-F(x;\Xi)\}}}. \tag{1.2}$$

Where Ξ is the vector of baseline distribution (BLDn) parameters.

This study aims to develop a new four parameter lifetime model labeled as ‘exponentiated alpha Weibull (EAW)’ based on the EPAIx family given by [24]. The proposed EAW model can provide several desirable properties and retains more flexibility in the shapes of HRFn, such as increasing, decreasing, constant, upside-down bathtub, bathtub, J, and reversed-J and S.

The rest of the manuscript is arranged as follows. In Section 2, the new sub-model EAW for four parameter based on the EPAIx family is proposed and graphically plotted to explore PDFn and CDFn. Reliability analysis and statistical properties of the EAW model are derived in Sections 3 and 4, respectively. Simulation is performed in Section 5. Estimation is provided in Section 6. Section 7 describes the empirical illustration using water runoff data. Finally, Section 8 concludes the study.

2. FOUR PARAMETER SUB-MODEL OF THE EPAIX GENERALIZED FAMILY OF DISTRIBUTIONS

In this section, a new sub-model is developed by considering the Weibull distribution (WDn) as the baseline model. The CDFn and PDFn of Weibull rv X with $\delta > 0$ as shape and $\gamma > 0$ as scale parameters are stated as follows:

$$F(x;\gamma,\delta) = 1 - e^{-\gamma x^{\delta}};\ x > 0, \tag{2.1}$$

and

$$f(x;\gamma,\delta) = \gamma\delta x^{\delta-1} e^{-\gamma x^{\delta}}; x > 0. \tag{2.2}$$

Using Eqs. (2.1) and (2.2) in Eqs. (1.1) and (1.2), a new sub-model of EPAIx generalized family of distributions given by [24] is obtained as follows:

The new four parameter exponentiated power alpha index-Weibull (or shortly, exponentiated alpha Weibull (EAW)) model has CDFn and PDFn as follows:

$$\Phi_{EAW}(x;\alpha,\theta,\gamma,\delta) = \left[\frac{\alpha^{\frac{\gamma x^{\delta}}{1+\gamma x^{\delta}}} - 1}{\alpha - 1}\right]^{\theta}; x > 0, \tag{2.3}$$

and

$$\phi_{EAW}(x;\alpha,\theta,\gamma,\delta) = \frac{\theta\gamma\delta\log\alpha}{\alpha-1}\frac{x^{\delta-1}}{[1+\gamma x^{\delta}]^2}\left[\frac{\alpha^{\frac{\gamma x^{\delta}}{1+\gamma x^{\delta}}} - 1}{\alpha - 1}\right]^{\theta-1} \alpha^{\frac{\gamma x^{\delta}}{1+\gamma x^{\delta}}};\ \alpha,\theta,\gamma,\delta > 0, \alpha \neq 1. \tag{2.4}$$

Graphically, PDFn and CDFn of four parameter EAW model for several parameter values are plotted in Figures 1-2.

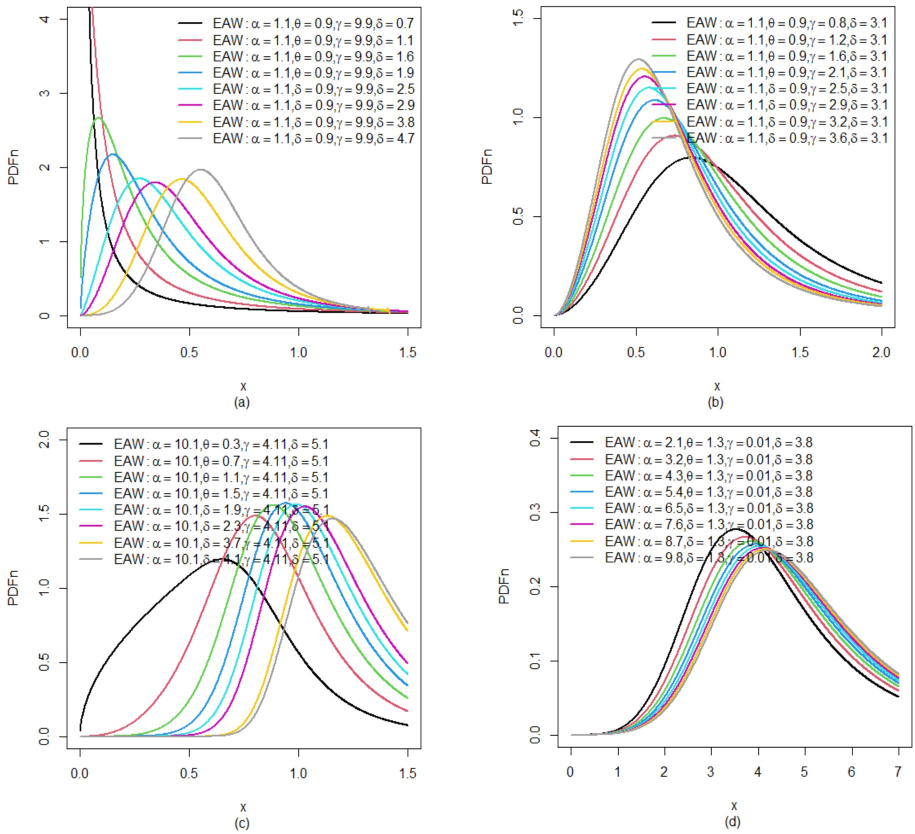
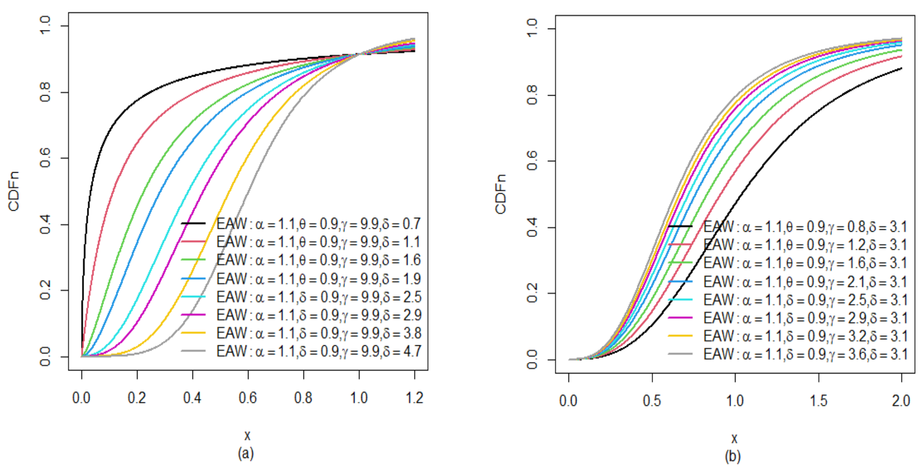


Figure 1. PDFn Graphs for Four Parameter EAW Distribution



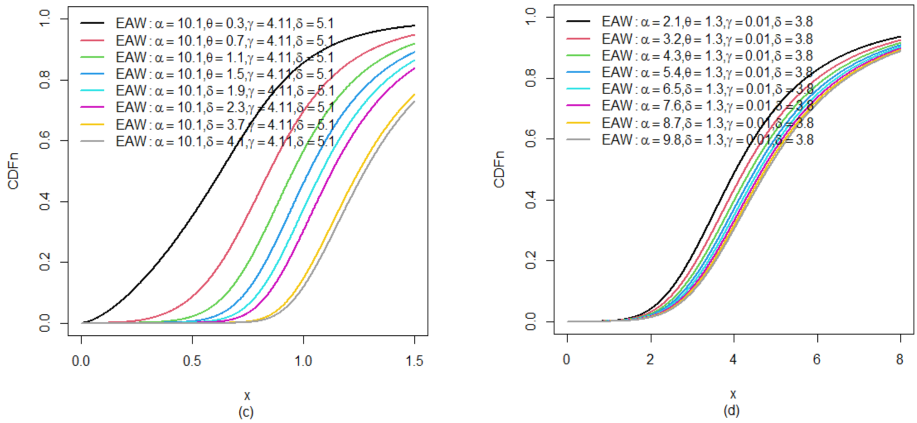


Figure 2. CDFn Graphs for Four Parameter EAW Distribution

3. RELIABILITY ANALYSIS (RA)

When t follows EAW $(\alpha, \theta, \gamma, \delta)$ model as lifetime rv, then CDFn and PDFn for reliability analysis are as follows:

$$\Phi_{EAW}(t; \alpha, \theta, \gamma, \delta) = \left[\frac{\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1}{\alpha - 1}\right]^{\theta} ; t > 0, \tag{3.1}$$

and

$$\phi_{EAW}(t; \alpha, \theta, \gamma, \delta) = \frac{\gamma\theta\delta \log \alpha}{\alpha - 1} \frac{t^{\delta-1}}{[1+\gamma t^{\delta}]^2} \left[\frac{\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1}{\alpha - 1}\right]^{\theta-1} \alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}}; t, \alpha, \theta, \gamma, \delta > 0, \alpha \neq 1. \tag{3.2}$$

In the following section, some reliability terms such as reliability function (RFn) R(t), HRFn h(t), cumulative HRFn (CHRFn) H(t), reversed HRFn (RHRFn) r(t), mean waiting time (MWTe), and mean residual life (MRLe) are defined

$$R_{EAW}(t) = 1 - \left[\frac{\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1}{\alpha - 1}\right]^{\theta}, \tag{3.3}$$

$$h_{EAW}(t) = \frac{\gamma\theta\delta \log\alpha \left[1+\gamma t^{\delta}\right]^{-2} t^{\delta-1} \alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} \left[\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1\right]^{\theta-1}}{(\alpha-1)^{\theta} - \left[\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1\right]^{\theta}}, \tag{3.4}$$

$$H_{EAW}(t) = -\log\left\{1 - \left[\frac{\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}} - 1}{\alpha - 1}\right]^{\theta}\right\}, \tag{3.5}$$

and

$$r_{EAW}(t) = \gamma\delta\theta \log\alpha \left[t^{\delta-1}\right]\left[1+\gamma t^{\delta}\right]^{-2}\left[1 - \alpha^{-\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}}\right]^{-1}. \tag{3.6}$$

Graphically RFn, HRFn, CHRFn, and RHRFn of four parameter EAW model for different parameter values are plotted in Figures 3-6.

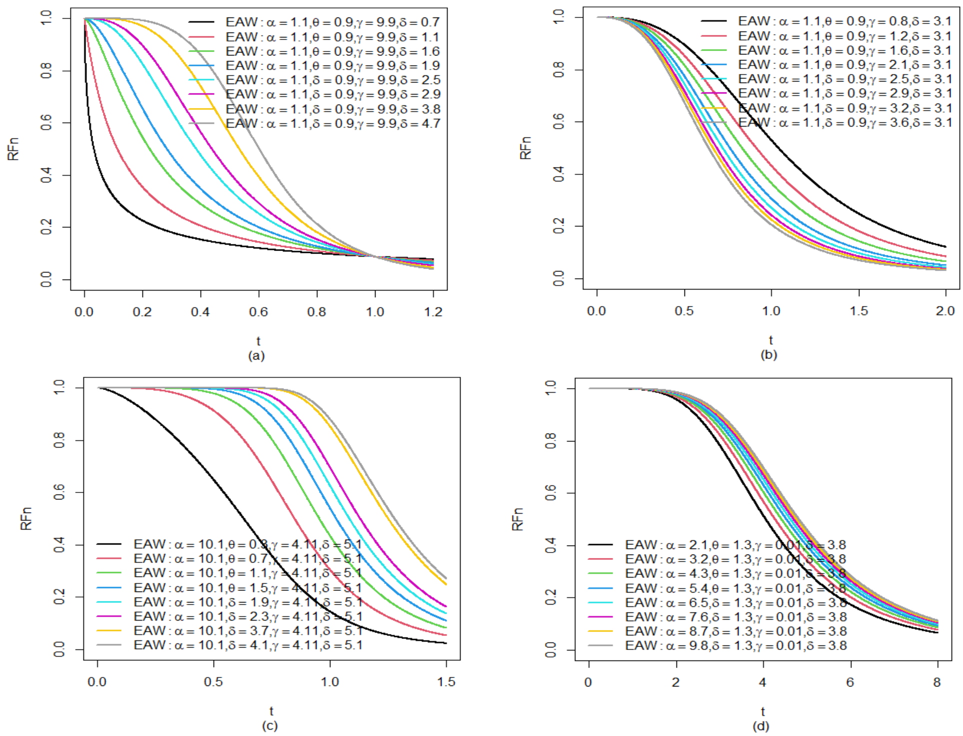


Figure 3. RFn Graphs for Four Parameter EAW Distribution.

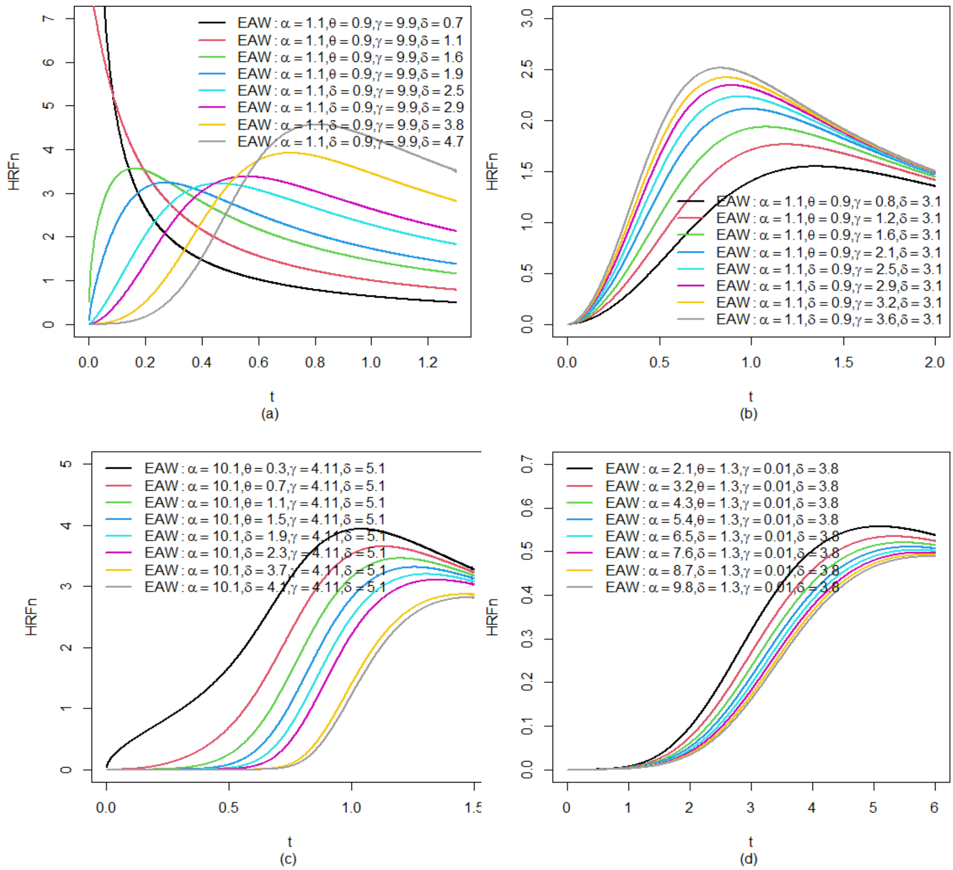
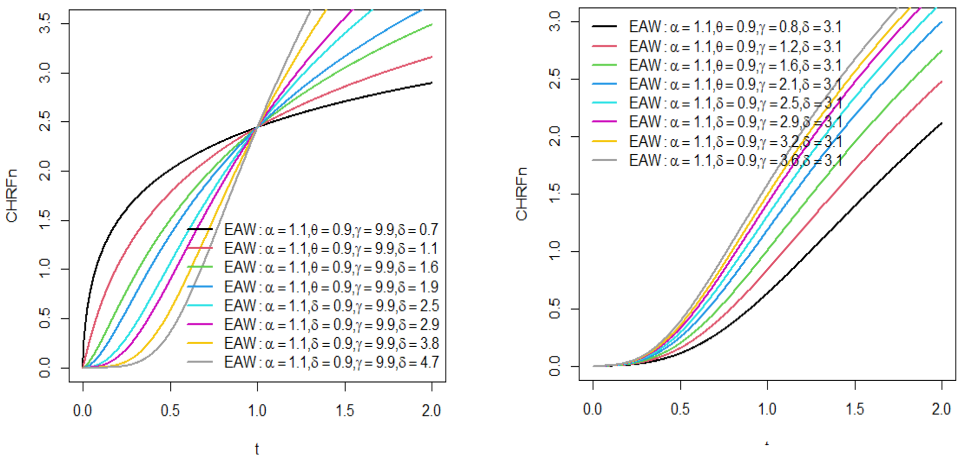


Figure 4. HRFn Graphs for Four Parameter EAW Distribution



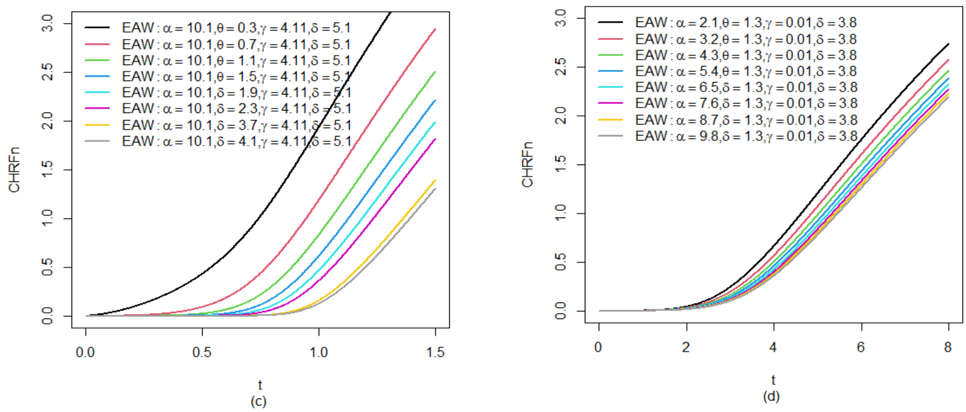


Figure 5. CHRFn Graphs for Four Parameter EAW Distribution

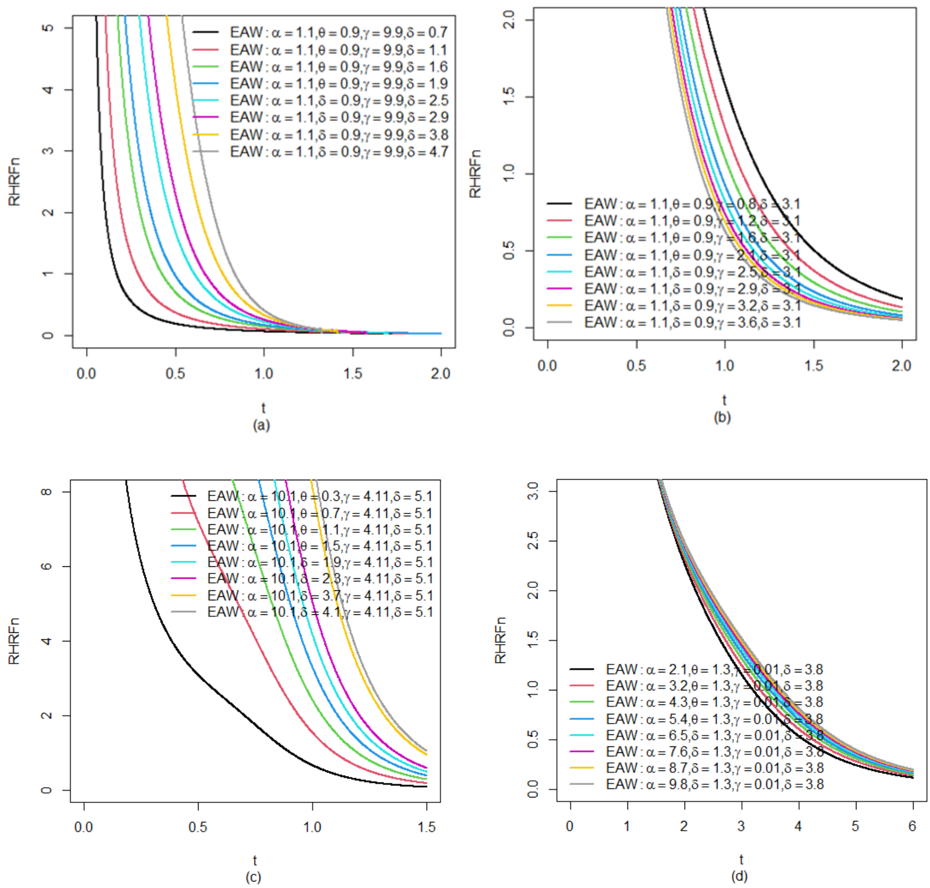


Figure 6. RHRFn Graphs for Four Parameter EAW Distribution.

3.1. Mean Waiting Time (MWTe)

The MWTe $\bar{M}(t)$ of rv t for four parameter EAW model is as follows:

$$\bar{\mathrm{M}}(\mathrm{t}) = \mathrm{t} - \left[\frac{1}{\Phi_{EAW}(t;\alpha,\theta,\gamma,\delta)}\int_0^t t\,\phi_{EAW}(t;\alpha,\theta,\gamma,\delta)dt\right]. \tag{3.7}$$

With generalized expansion of binomial, we have

$$(1-Y)^B = \sum_{k=0}^{\infty}(-1)^k \binom{B}{k} Y^k. \tag{3.8}$$

Eq. (3.8) holds for $|Y| < 1$ and B is arbitrary real non-integer. Also, the series of Maclaurin is as follows:

$$\alpha^{By} = \sum_{j=0}^{\infty}\frac{(\log\alpha)^j (By)^j}{j!}. \tag{3.9}$$

By combining Eqs. (3.2), (3.8), and (3.9), we obtain

$$\int_0^t t\,\phi_{EAW}(t)dt = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta-1}{k}\frac{\theta(-1)^{\theta-1+k}(\log\alpha)^{j+1}(k+1)^j}{j!\,\gamma^{\frac{1}{\delta}}(\alpha-1)^{\theta}}B_u\left(\left(j+\frac{1}{\delta}+1\right),\left(1-\frac{1}{\delta}\right)\right). \tag{3.10}$$

Substituting Eqs. (3.1) and (3.10) in Eq. (3.7), we obtain MWTe as

$$\bar{\mathrm{M}}(\mathrm{t}) = t - \theta\left[\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}}-1\right]^{-\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\binom{\theta-1}{k}(-1)^{\theta-1+k}(k+1)^j}{j!\,\gamma^{\frac{1}{\delta}}(\log\alpha)^{-(j+1)}}B_u\left(\left(j+\frac{1}{\delta}+1\right),\left(1-\frac{1}{\delta}\right)\right). \tag{3.11}$$

3.2. Mean Residual Life (MRLe)

The MRLe M(t) of rv t for four parameter EAW model is as follows:

$$M(t) = \frac{1}{R_{EAW}(t)}\int_t^{\infty} t\,\phi_{EAW}(t;\alpha,\theta,\gamma,\delta)dt - t \tag{3.12}$$

or

$$M(t)=\frac{1}{R_{EAW}(t)}\left[E_{EAW}(t)-\int_0^t t\,\phi_{EAW}(t;\alpha,\theta,\gamma,\delta)dt\right]-t. \qquad (3.13)$$

The integral used in Eq. (3.13) was obtained in Eq. (3.10). Now, we obtain $E_{EAW}(t)$ in the following as

$$E_{EAW}(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{\theta-1}{j}\frac{\theta(-1)^{\theta-1+j}(\log\alpha)^{k+1}(j+1)^k}{k!\;\gamma^{\frac{1}{\delta}}(\alpha-1)^{\theta}}B\left(\left(k+\frac{1}{\delta}+1\right),\left(1-\frac{1}{\delta}\right)\right). \qquad (3.14)$$

Using Eqs. (3.3), (3.10), and (3.14) in Eq. (3.13), we obtain the MRLe as

$$M(t)=\frac{\theta\,\gamma^{-\frac{1}{\delta}}}{(\alpha-1)^{\theta}-\left[\alpha^{\frac{\gamma t^{\delta}}{1+\gamma t^{\delta}}}-1\right]^{\theta}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left\{\binom{\theta-1}{j}\frac{(\log\alpha)^{k+1}(-1)^{\theta-1+j}(j+1)^k}{k!}B\left(\left(k+\frac{1}{\delta}+1\right),\left(1-\frac{1}{\delta}\right)\right)-\binom{\theta-1}{k}\frac{(\log\alpha)^{j+1}(-1)^{\theta-1+k}(k+1)^j}{j!}B_u\left(\left(j+\frac{1}{\delta}+1\right),\left(1-\frac{1}{\delta}\right)\right)\right\}-t, \qquad (3.15)$$

Where $u=\gamma t^{\delta}$, $B(b_1,b_2)=\int_0^{\infty}\frac{u^{b_1-1}}{(1+u)^{b_1+b_2}}du$ beta function (BFn), and $B_u\,(b_1+b_2)=\int_0^{u}\frac{u^{b_1-1}}{(1+u)^{b_1+b_2}}du$ is incomplete BFn.

4. STATISTICAL PROPERTIES

In this section, some statistical properties of four parameter EAW model are explored.

4.1. Quantile Function (QFn) and Generation of Random Numbers

With Eq. (2.3), we obtain pth quantile of the four parameter EAW model as follows:

$$\left[\frac{\alpha^{\frac{\gamma x_p^\delta}{1+\gamma x_p^\delta}}-1}{\alpha-1}\right]^\theta=p,$$

After solving the above, we obtain QFn x_p as

$$x_p=\frac{1}{\gamma^{\frac{1}{\delta}}}\left[\frac{\log\left\{1-(1-\alpha)p^{\frac{1}{\theta}}\right\}}{\log\alpha-\log\left\{1-(1-\alpha)p^{\frac{1}{\theta}}\right\}}\right]^{\frac{1}{\delta}};0<p<1. \quad (4.1)$$

Setting $p=0.25,0.50,0.75$ in Eq. (4.1), we obtain lower quartile $x_{0.25}$, median $x_{0.50}$, and upper quartile $x_{0.75}$. The inversion method with Eq. (4.2) provides the random numbers when u follows a uniform distribution,

$$x=\frac{1}{\gamma^{\frac{1}{\delta}}}\left[\frac{\log\left\{1-(1-\alpha)u^{\frac{1}{\theta}}\right\}}{\log\alpha-\log\left\{1-(1-\alpha)u^{\frac{1}{\theta}}\right\}}\right]^{\frac{1}{\delta}};u\sim U(0,1). \quad (4.2)$$

Table 1 presents the points of percentage value of the four parameter EAW model for specific parameters values.

Table. 1. Skewness, kurtosis, and points of percentage value for some values of α,θ,γ, and δ for four parameter EAWmodel.

			$\alpha=1.1,\theta=0.9$			
	δ	Skewnes	Kurtosis	Q_1	$Median$	Q_3
$\gamma=9.9$	0.7	6.79028	44.40655	0.012930	0.04763	0.179080
	1.1	3.25922	13.39244	0.055198	0.09656	0.338678
	1.6	3.28720	10.07080	0.106750	0.22957	0.359300
	1.9	1.07105	0.33021	0.187920	0.30873	0.551780
	2.5	1.55706	1.30684	0.249800	0.34106	0.523100
	2.9	2.19435	5.65788	0.238100	0.33369	0.495400
	3.8	1.99001	3.56985	0.408100	0.51293	0.745700
	4.7	1.25995	1.27538	0.461500	0.57519	0.778300
			$\alpha=1.1,\theta=0.9$			
	γ	Skewnes	Kurtosis	Q_1	$Median$	Q_3
$\delta=3.1$	0.8	2.04456	6.42292	0.803600	1.05115	1.641100
	1.2	1.59346	2.86228	0.539200	0.86536	1.222500
	1.6	2.76220	8.50941	0.662100	0.89902	1.158500

	2.1	0.92640	0.77347	0.612800	0.82887	1.120400
	2.5	1.05504	0.21845	0.545700	0.75567	1.200300
	2.9	1.32920	2.90431	0.567700	0.69719	0.961100
	3.2	0.61375	0.12693	0.479100	0.66713	0.912200
	3.6	4.59271	23.17074	0.411100	0.62944	0.917100
$\alpha = 10.1, \gamma = 4.11$						
	θ	Skewnes	Kurtosis	Q_1	$Median$	Q_3
$\delta = 5.1$	0.3	0.20394	-0.63173	0.496300	0.65046	0.928500
	0.7	0.96743	0.80922	0.509200	0.75718	1.025800
	1.1	0.81207	0.63018	0.774400	0.94208	1.207000
	1.5	4.66601	24.73251	0.819900	0.98435	1.133500
	1.9	2.02736	5.68857	0.931300	1.08535	1.249800
	2.3	0.54179	-0.37135	0.968300	1.12266	1.326400
	3.7	0.74986	0.07177	1.092500	1.18973	1.465000
	4.1	1.15742	0.82227	1.073800	1.19493	1.538400
$\theta = 1.3, \gamma = 0.01$						
	α	Skewnes	Kurtosis	Q_1	$Median$	Q_3
$\delta = 3.8$	2.1	2.19925	6.81219	3.323000	3.99854	5.336000
	3.2	0.61693	-0.15077	3.284000	4.03830	5.500000
	4.3	1.63138	3.14881	3.061000	4.29217	5.497000
	5.4	0.75978	-0.06059	3.432000	4.24603	5.684000
	6.5	1.12352	0.58028	3.649000	4.43350	5.713000
	7.6	0.83245	0.52685	3.349000	4.46603	6.109000
	8.7	0.96343	0.77240	3.956000	4.96290	6.997000
	9.8	1.13100	1.88915	3.640000	5.12035	6.964000

4.2. Skewness and Kurtosis on QFn

Skewness and kurtosis measure the effect of shape parameters. Here, measures of skewness and kurtosis given by [25, 26] are calculated for the proposed EAW model.

$$S_{Sk} = \frac{Q_{0.75} + Q_{0.25} - 2Q_{0.50}}{Q_{0.75} - Q_{0.25}},$$

and

$$K_{Ku} = \frac{Q_{0.375} + Q_{0.125} + Q_{0.875} - Q_{0.625}}{Q_{0.750} - Q_{0.125}},$$

Where $Q_{(.)}$ is QFn.

Graphically, skewness and kurtosis of four parameter EAW model for particular parameters values are plotted in Figures 7-8. In Figures 7-8, "A" stands for "α", "T" stands for "θ", "g" stands for "γ", and "D" stands for "δ".

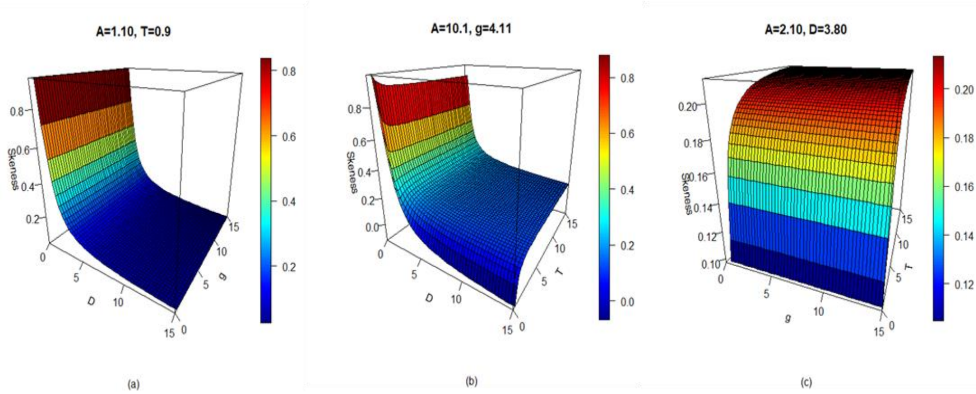


Figure 7. Plots for Skewness of EAW $(\alpha, \theta, \gamma, \delta)$ **(a)** as function of $\delta$and γ for specific α, θ **(b)** as function of δ and θ for specific α, γ **(c)** as function of γ and θ for specific α, δ.

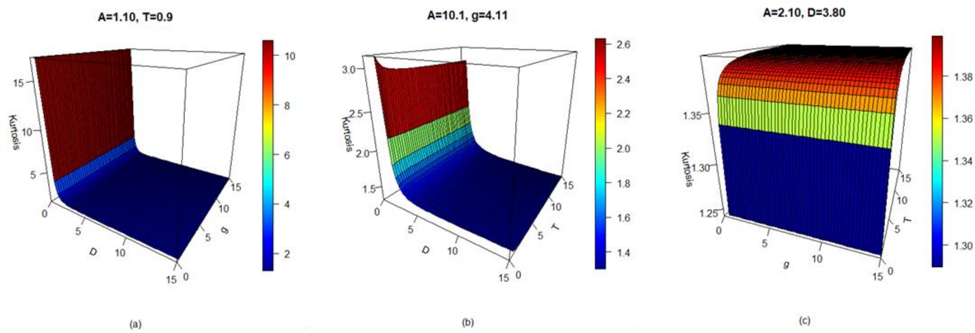


Figure 8. Plots for Kurtosis of EAW $(\alpha, \theta, \gamma, \delta)$ **(a)** as function of δ and γ for specific α, θ **(b)** as function of δ and θ for specific α, γ **(c)** as function of γ and θ for specific α, δ.

4.3. Mode

To obtain the mode of the EAW model using Eq. (2.4), we differentiate the natural logarithm of $\phi_{EAW}\,(x; \alpha, \theta, \gamma, \delta)$ and equate it to zero, that is, $\frac{d}{dx}\ln\{\phi_{EAW}\,(x; \alpha, \theta, \gamma, \delta)\} = 0$. Afterwards, we get

$$\gamma\delta\log\alpha\frac{x^{\delta-1}}{(1+\gamma x^{\delta})^2}\left\{\frac{\theta-1}{1-\alpha^{-\frac{\gamma x^{\delta}}{1+\gamma x^{\delta}}}}+1\right\}-2\gamma\delta\frac{x^{\delta-1}}{1+\gamma x^{\delta}}+\frac{\delta-1}{x}=0. \tag{4.3}$$

According to x, it is difficult to find out the analytical solution (ASn) of Eq. (4.3) because it is a nonlinear equation (NLEq). So, a numerical solution (NSn) is required and x_0 is the root of Eq. (4.3), if $\frac{d^2}{dx^2}\ln\{\phi_{EAW}\ (x;\alpha,\theta,\gamma,\delta)\}<0$ exist.

4.4. Incomplete Moments (IMs)

If rv X follows EAW $(\alpha,\theta,\gamma,\delta)$ withPDFn in Eq. (2.4), then nth IMs of four parameter EAW model is as follows:

$$M_n(x)=\int_{-\infty}^{x}x^n\,\phi_{EAW}(x;\alpha,\theta,\gamma,\delta)dx, \tag{4.4}$$

With the help of Eqs. (2.4), (3.8), (3.9) and (4.4), we obtain

$$M_n(x)=\frac{\theta}{\gamma^{\frac{n}{\delta}}(\alpha-1)^{\theta}}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\frac{(-1)^{i+\theta-1}(\log\alpha)^{1+l}}{l!\ (1+i)^{-l}}B_t\left(\left(1+l+\frac{n}{\delta}\right),\left(1-\frac{n}{\delta}\right)\right). \tag{4.5}$$

Where $t=\gamma x^{\delta}$ and $\int_0^t\frac{t^{l+\frac{n}{\delta}}}{(1+t)^{l+2}}dt=B_t\left(\left(1+l+\frac{n}{\delta}\right),\left(1-\frac{n}{\delta}\right)\right)$ is incomplete BFn.

For $n=1$, Eq. (4.5) becomes first IM which is as follows:

$$M_1(x)=\frac{\theta}{\gamma^{\frac{1}{\delta}}(\alpha-1)^{\theta}}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\frac{(-1)^{i+\theta-1}(\log\alpha)^{1+l}}{l!\ (1+i)^{-l}}B_t\left(\left(1+l+\frac{1}{\delta}\right),\left(1-\frac{1}{\delta}\right)\right). \tag{4.6}$$

4.5. Moments

About origin, the qth raw moments of rv X that follows EAW $(\alpha,\theta,\gamma,\delta)$ having PDFn in Eq. (2.4) is as follows:

$$\mu_q'(x) = \int_{-\infty}^{\infty} x^q \, \phi_{EAW}(x;\alpha,\theta,\gamma,\delta)dx, \tag{4.7}$$

Combining Eq. (2.4) and Eq. (4.7) and using Eqs. (3.8) and (3.9), we obtain

$$\mu_q'(x) = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\binom{\theta-1}{j}\frac{\theta(-1)^{j+\theta-1}(\log\alpha)^{1+m}}{m!\,\gamma^{\frac{q}{\delta}}(\alpha-1)^{\theta}(1+j)^{-m}}\,B\left(\left(1+m+\frac{q}{\delta}\right),\left(1-\frac{q}{\delta}\right)\right). \tag{4.8}$$

Where $\int_0^\infty \frac{t^{m+\frac{q}{\delta}}}{(t+1)^{m+2}}dt = B\left(\left(1+m+\frac{q}{\delta}\right),\left(1-\frac{q}{\delta}\right)\right)$ is BFn and $t=\gamma x^{\delta}$.

For q = 1, 2, 3, and 4 in Eq (4.8), we get four raw moments $\mu_1'(x)$, $\mu_2'(x), \mu_3'(x)$, and $\mu_4'(x)$.

4.6. Moment Generating Function (MGFn)

With Eq. (2.4), the MGFn of rv X that follows four parameter EAW model is as follows:

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\,\phi_{EAW}(x;\alpha,\theta,\gamma,\delta)dx, \tag{4.9}$$

Using series $e^{ty} = \sum_{l=0}^{\infty}\frac{(ty)^l}{l!}$, Eqs. (3.8) and (3.9) in Eq. (4.9), we obtain

$$M_X(t) = \sum_{p,q,m=0}^{\infty}\binom{\theta-1}{m}\frac{\theta t^p(\log\alpha)^{1+q}(-1)^{m+\theta-1}}{\gamma^{\frac{p}{\delta}}p!\,q!\,(\alpha-1)^{\theta}(m+1)^{-q}}B\left(\left(\frac{p}{\delta}+q+1\right),\left(1-\frac{p}{\delta}\right)\right). \tag{4.10}$$

Where $\int_0^\infty \frac{t^{q+\frac{p}{\delta}}}{(1+t)^{q+2}}dt = B\left(\left(\frac{p}{\delta}+q+1\right),\left(1-\frac{p}{\delta}\right)\right)$ is BFn and $u=\gamma x^{\delta}$.

4.7. Mean Deviation (MDn)

About mean (μ_1') and median (M), the MDn of rv X for four parameter EAW model having CDFn in Eq. (2.3) and PDFn in Eq. (2.4) can be obtained by using the following formulas,

$$MDn_{\mu_1'}(x) = 2\mu_1'\Phi_{EAW}(\mu_1') - 2M_1(\mu_1'), \tag{4.11}$$

$$MDn_M(x) = \mu_1' - 2M_1(M). \tag{4.12}$$

The Eqs. (4.11) and (4.12) can be obtained by substituting μ_1' from Eq. (4.8) at q = 1, $\Phi_{EAW}(x)$ evaluated at μ_1' from Eq. (2.3) and $M_1\ (x)$evaluated at μ_1' and M from Eq. (4.1) at $p = 0.50$.

4.8. Probability Weighted Moments (PWMs)

The PWMs of rv X for four parameter EAW model are obtained as follows:

$$\tau_{m_1,m_2} = E[X^{m_1}\Phi_{EAW}(x;\alpha,\theta,\gamma,\delta)^{m_2}]$$
$$= \int_{-\infty}^{\infty} x^{m_1}[\Phi_{EAW}(x;\alpha,\theta,\gamma,\delta)]^{m_2}\phi_{EAW}(x;\alpha,\theta,\gamma,\delta)dx, \tag{4.13}$$

Using Eqs. (2.3), (2.4), (3.8), and (3.9) in Eq. (4.13), we obtain

$$\tau_{m_1,m_2} = \sum_{m_3=0}^{\infty}\sum_{m_4=0}^{\infty}\binom{(m_2+1)\theta-1}{m_3}\frac{\theta(\log\alpha)^{m_4+1}(-1)^{(m_2+1)\theta-1+m_3}}{m_4!\,\gamma^{\frac{m_1}{\delta}}(\alpha-1)^{(m_2+1)\theta}(m_3+1)^{-m_4}}B\left(\left(\frac{m_1}{\delta}+m_4+1\right),\left(1-\frac{m_1}{\delta}\right)\right). \tag{4.14}$$

Where $B\left(\left(\frac{m_1}{\delta}+m_4+1\right),\left(1-\frac{m_1}{\delta}\right)\right) = \int_0^\infty \frac{v^{\frac{m_1}{\delta}+m_4}}{(1+v)^{m_4+2}}dv$ is BFn and $v = \gamma x^{\delta}$.

4.9. Rényi Entropy (REp)

[27] measured the uncertainty of variation by developing the following entropy measure,

$$R^c(x) = \frac{1}{1-c}\log\int_{-\infty}^{\infty}[\phi_{EAW}(x;\alpha,\theta,\gamma,\delta)]^c dx;\ c \neq 1, c > 0, \tag{4.15}$$

For rv X that follows four parameter EAW model, the REp using Eqs. (2.4), (3.8), and (3.9) in Eq. (4.15) is as follows:

$$R^c(x) = \frac{1}{1-c}\log\left[\begin{array}{l}\displaystyle\sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\frac{\theta^c\delta^{c-1}(\log\alpha)^{h_2+c}\,(-1)^{h_1+c(\theta-1)}}{h_2!\,(\alpha-1)^{c\theta}(h_1+c)^{-h_2}\gamma^{\frac{1}{\delta}(1-c)}}\binom{c(\theta-1)}{h_1} \\ \displaystyle\times B\left(\left(c+(1-c)\frac{1}{\delta}+h_2-1\right),\left(1+c-(1-c)\frac{1}{\delta}\right)\right)\end{array}\right]. \quad (4.16)$$

Here, $w=\gamma x^{\delta}$ and the BFn used in Eq. (5.16) is

$$\int_0^{\infty}\frac{w^{c+(1-c)\frac{1}{\delta}+h_2-1}}{(1+w)^{2c+h_2}}dw = B\left(\left(c+(1-c)\frac{1}{\delta}+h_2-1\right),\left(1+c-(1-c)\frac{1}{\delta}\right)\right).$$

4.10. Order Statistic (OSt)

Suppose $X_1 < X_2 < \cdots < X_s$ are random samples chosen from four parameter EAW model having CDFn $\Phi_X(x;\alpha,\theta,\gamma,\delta)$ in Eq. (2.3) and PDFn $\phi_X(x;\alpha,\theta,\gamma,\delta)$ in Eq. (2.4), while $X_{1:s} \le X_{2:s} \le \cdots \le X_{s:s}$ are order statistic, then PDFn of $X_{\tau:s}; \tau = 1,2,\cdot\cdot\cdot,s$ is as follows:

$$\phi_{X_{\tau:s}}(x;\alpha,\theta,\gamma,\delta) = \frac{s!}{(s-\tau)!\,(\tau-1)!}[\Phi_X(x;\alpha,\theta,\gamma,\delta)]^{\tau-1}[1-\Phi_X(x;\alpha,\theta,\gamma,\delta)]^{s-\tau}\phi_X(x;\alpha,\theta,\gamma,\delta). \quad (4.17)$$

With Eqs. (2.3) and (2.4) in Eq. (4.17) and using Eqs. (3.8) and (3.9), we obtain the PDFn $\phi_{X_{\tau:s}}(x;\alpha,\theta,\gamma,\delta)$ of τth OSt for four parameter EAW model as follows:

$$\phi_{X_{\tau:s}}(x;\alpha,\theta,\gamma,\delta) = \sum_{i=0}^{s-\tau}\sum_{j=0}^{\theta(\tau+i)-1}\sum_{k=0}^{\infty}\frac{s!\theta\delta\gamma^{k+1}(\log\alpha)^{k+1}(-1)^{i+j+\theta(\tau+i)-1}(1+j)^k}{k!(s-\tau)!(\tau-1)!(\alpha-1)^{\theta(\tau+i)}}\times \binom{s-\tau}{i}\binom{\theta(\tau+i)-1}{j}\frac{x^{\delta(k+1)-1}}{(1+\gamma x^{\delta})^{k+2}}. \quad (4.18)$$

4.11. Moment of Order Statistic

The nth moment of τth OSt for four parameter EAW model is defined as follows:

$$E(X_{\tau:s}^{n}) = \int_{-\infty}^{\infty} x_{\tau:s}^{n}\, \phi_{X_{\tau:s}}(x;\alpha,\theta,\gamma,\delta)dx_{\tau:s}, \tag{4.19}$$

Using Eq. (4.18) in Eq. (4.19) and solving the integral by BFn with $y = \gamma x_{\tau:s}^{\delta}$, we have

$$E(X_{\tau:s}^{n}) = \sum_{i=0}^{s-\tau} \sum_{j=0}^{\theta(\tau+i)-1} \sum_{k=0}^{\infty} \frac{s!\,\theta(\log\alpha)^{k+1}(-1)^{i+j+\theta(\tau+i)-1}(1+j)^{k}}{(s-\tau)!\,(\tau-1)!\,k!\,\gamma^{\frac{n}{\delta}}(\alpha-1)^{\theta(\tau+i)}} \binom{s-\tau}{i} \binom{\theta(\tau+i)-1}{j} B\left(\left(\frac{n}{\delta} + k + 1\right), \left(1-\frac{n}{\delta}\right)\right). \tag{4.20}$$

5. SIMULATION STUDY

This section provides the simulation study for maximum likelihood estimates (MLEs) of four parameter EAW model. To observe the performance of MLEs, N = 1,000 samples from EAW $(\alpha,\theta,\gamma,\delta)$ model are generated for different values of $(\alpha,\theta,\gamma,\delta)$ under sample sizes $n = 25, 50, \cdot\cdot\cdot, 250.$ Moreover, we compute the average estimates (AEs), absolute average biases (ABs), and mean square errors (MSEs). These simulated results are presented in Table 2.

As expected, as n increases the AEs move to actual values and the absolute ABs and MSEs tend towards zero. These results are also plotted graphically in Figures 9-10. In this simulation study, N = 1,000 samples from four parameter EAW model are generated under sample sizes$n$ = 25, 50, …, 250 with actual values of parameter $\alpha = 1.10, \theta = 0.90, \gamma = 9.90, \delta = 1.60,$ and $\alpha = 1.11, \theta = 0.79, \gamma = 2.78, \delta = 3.80.$ The curves of the absolute ABs and MSEs are plotted in Figures 9-10 for estimates. These figures project absolute ABs and MSEs curves approach quickly to the x-axis.

Table 2. Absolute ABs and MSEs Corresponding to AEs of Four Parameter EAW Model

		Set I			Set II		
		$\alpha = 1.10, \theta = 0.90, \gamma = 9.90, \delta = 1.60$			$\alpha = 1.11, \theta = 0.79, \gamma = 2.78, \delta = 3.8$		
N	Parameters	AEs	Abs. ABs	MSEs	AEs	Abs. Abs	MSEs
25	$\hat{\alpha}$	1.096014	0.003986	0.051683	1.342813	0.232813	0.883443
	$\hat{\theta}$	0.858815	0.041185	0.014732	0.759283	0.030717	0.023779
	$\hat{\gamma}$	4.924908	4.975092	24.94871	2.663724	0.116276	0.273707
	$\hat{\delta}$	1.627597	0.027597	0.086284	3.966468	0.166468	0.206193

		Set I			Set II		
		$\alpha=1.10,\theta=0.90,\gamma=9.90,\delta=1.60$			$\alpha=1.11,\theta=0.79,\gamma=2.78,\delta=3.8$		
50	$\hat{\alpha}$	1.085719	0.014281	0.003826	1.199112	0.089112	0.299091
	$\hat{\theta}$	0.879994	0.020006	0.003090	0.763167	0.026833	0.009827
	$\hat{\gamma}$	4.988885	4.911115	24.12604	2.709362	0.070638	0.116479
	$\hat{\delta}$	1.588484	0.011516	0.004176	3.893479	0.093479	0.103996
75	$\hat{\alpha}$	1.089100	0.010900	0.001179	1.119575	0.009575	0.027345
	$\hat{\theta}$	0.887661	0.012339	0.001831	0.783988	0.006012	0.002351
	$\hat{\gamma}$	4.994570	4.905430	24.06543	2.764667	0.015333	0.023527
	$\hat{\delta}$	1.592067	0.007933	0.004113	3.821267	0.021267	0.019885
100	$\hat{\alpha}$	1.092906	0.007094	0.000646	1.119948	0.009948	0.025376
	$\hat{\theta}$	0.893518	0.006482	0.000667	0.784188	0.005812	0.001726
	$\hat{\gamma}$	4.995993	4.904007	24.05019	2.765108	0.014892	0.019440
	$\hat{\delta}$	1.592834	0.007166	0.000811	3.819476	0.019476	0.017068
125	$\hat{\alpha}$	1.094379	0.005621	0.000507	1.111372	0.001372	0.002748
	$\hat{\theta}$	0.893608	0.006392	0.001079	0.787662	0.002338	0.001053
	$\hat{\gamma}$	4.996814	4.903186	24.04187	2.772352	0.007648	0.008754
	$\hat{\delta}$	1.596637	0.003363	0.002741	3.812151	0.012151	0.011152
150	$\hat{\alpha}$	1.096559	0.003441	0.000308	1.113073	0.003073	0.007074
	$\hat{\theta}$	0.896658	0.003342	0.000318	0.788953	0.001047	0.000311
	$\hat{\gamma}$	4.998369	4.901631	24.02642	2.778206	0.001794	0.002215
	$\hat{\delta}$	1.596807	0.003193	0.000397	3.804372	0.004372	0.002952
175	$\hat{\alpha}$	1.097989	0.002011	0.000179	1.109915	0.000085	0.000294
	$\hat{\theta}$	0.898081	0.001919	0.000168	0.789771	0.000229	0.000080
	$\hat{\gamma}$	4.999202	4.900798	24.01798	2.779943	0.000057	0.000293
	$\hat{\delta}$	1.598326	0.001674	0.000151	3.801421	0.001421	0.001146
200	$\hat{\alpha}$	1.098523	0.001477	0.000137	1.109954	0.000046	0.000055
	$\hat{\theta}$	0.898502	0.001498	0.000161	0.789712	0.000288	0.000023
	$\hat{\gamma}$	4.999841	4.900159	24.01157	2.779838	0.000162	0.000079
	$\hat{\delta}$	1.598500	0.001500	0.000149	3.800324	0.000324	0.000066
225	$\hat{\alpha}$	1.099181	0.000819	0.000077	1.109701	0.000299	0.000023
	$\hat{\theta}$	0.899299	0.000701	0.000056	0.789806	0.000194	0.000014
	$\hat{\gamma}$	5.000000	4.900000	24.01000	2.779994	0.000006	0.000016
	$\hat{\delta}$	1.599086	0.000914	0.000098	3.800064	0.000064	0.000050
250	$\hat{\alpha}$	1.099313	0.000687	0.000061	1.110196	0.000196	0.000072
	$\hat{\theta}$	0.899350	0.000650	0.000057	0.789819	0.000181	0.000017
	$\hat{\gamma}$	4.999666	4.900334	24.01338	2.779717	0.000283	0.000041
	$\hat{\delta}$	1.599293	0.000707	0.000069	3.800391	0.000391	0.000322

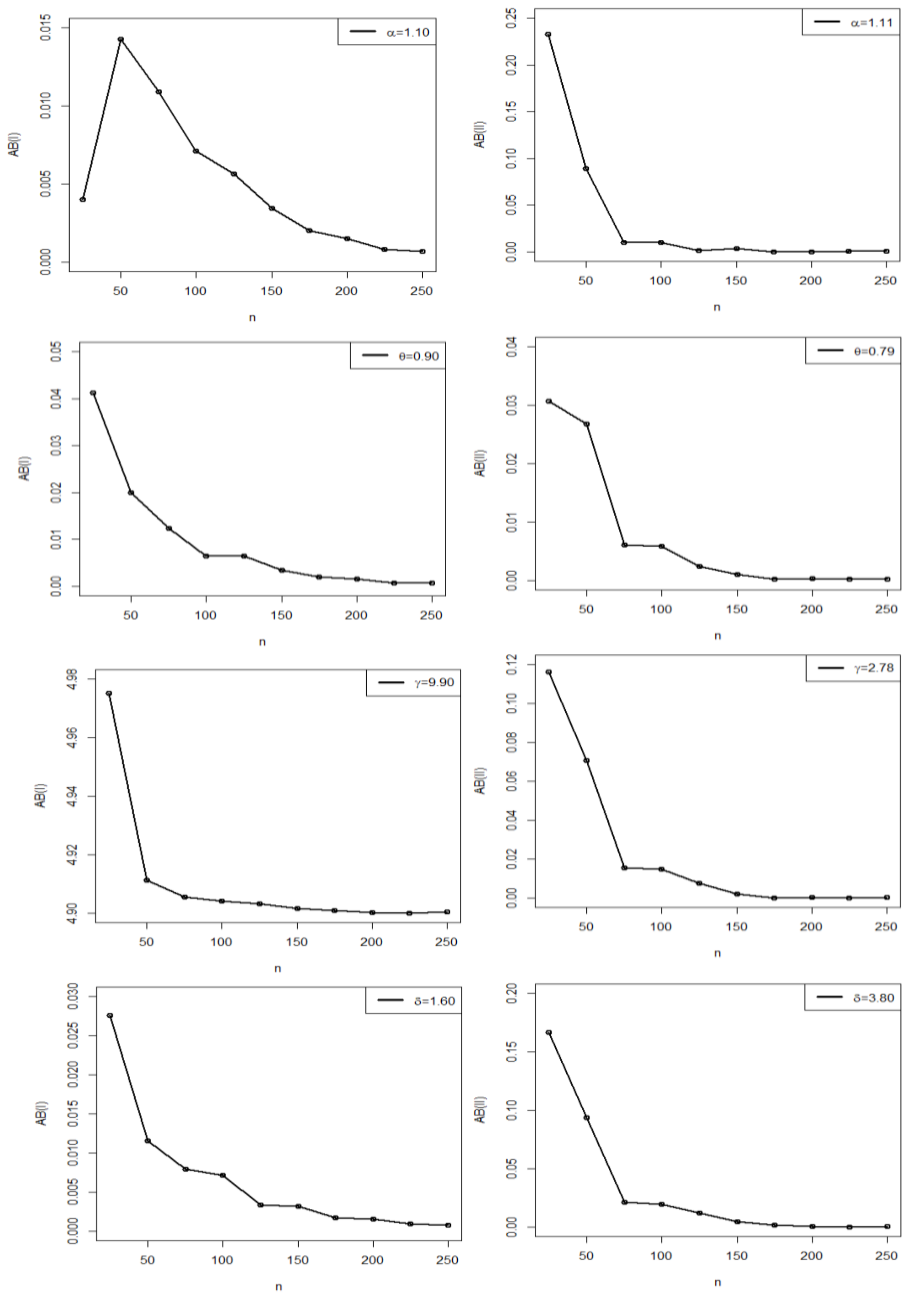


FigURE 9. Plots for Absolute ABs of EAW $(\alpha, \theta, \gamma, \delta)$ Model for Sets I (1.10, 0.90, 9.90, 1.60) and II (1.11, 0.79,2.78, 3.80).

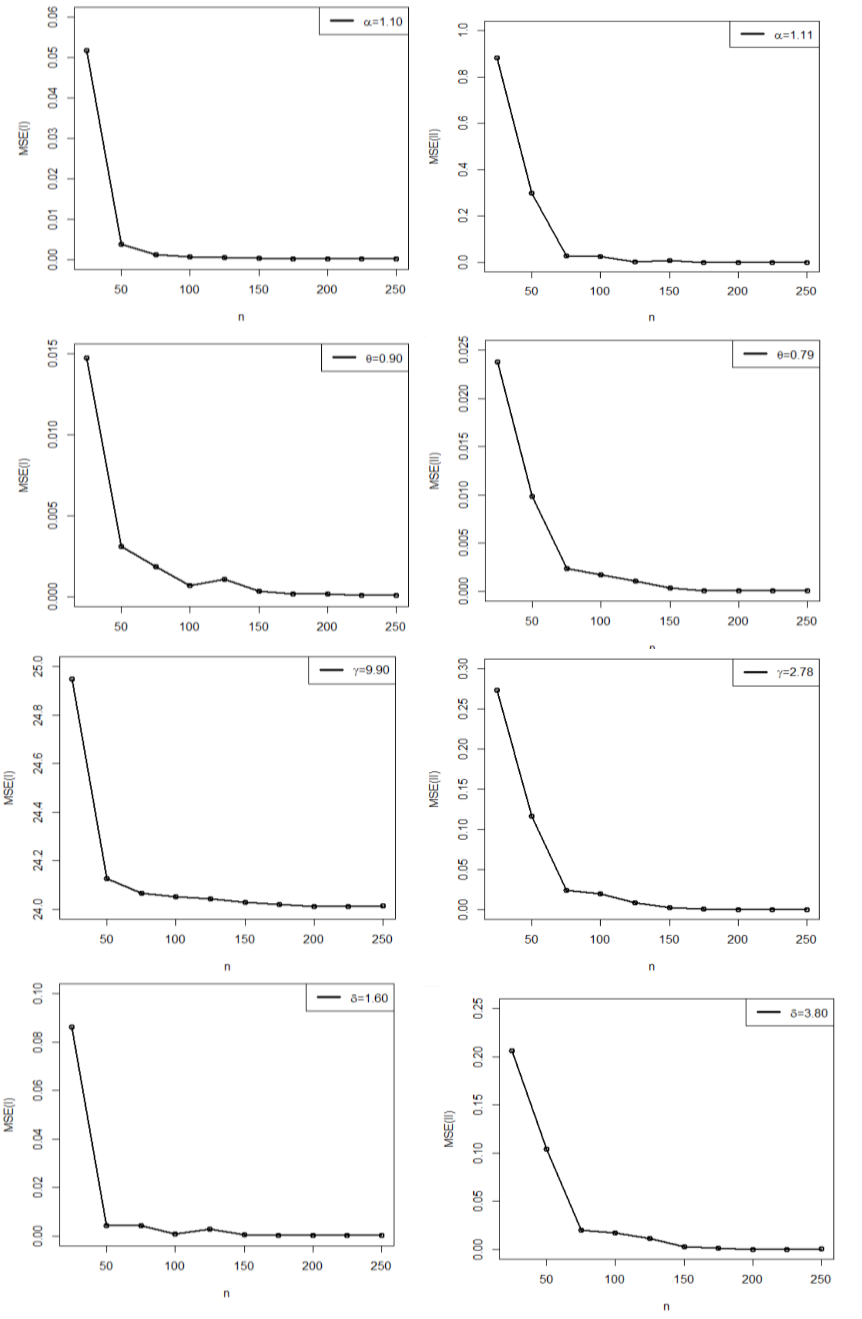


Figure 10. Plots for MSEs of EAW $(\alpha, \theta, \gamma, \delta)$ Model for Sets I (1.10, 0.90, 9.90, 1.60) and II (1.11,0.79,2.78, 3.80).

6. ESTIMATION

Supposed $x_1, x_2, \cdot\cdot\cdot, x_n$ sample be chosen from four parameter EAW model with PDFn developed in Eq. (2.4) for the vector of parameters $\upsilon = (\alpha, \theta, \gamma, \delta)^T$. For MLEs, we write an expression of log-likelihood function (LLFn) for υ according to maximum likelihood (ML) method as follows:

$$l = l(\upsilon)$$

$$= n\log\theta + n\log\gamma + n\,\log\delta + n\log(\log\alpha) - n\,\theta\log(\alpha-1) + (\delta-1)\sum_{i=1}^{n}\log x_i$$

$$- 2\sum_{i=1}^{n}\log\left[1+\gamma\, x_i^{\delta}\right] + (\theta$$

$$- 1)\sum_{i=1}^{n}\log\left[\alpha^{\frac{\gamma\, x_i^{\delta}}{1+\gamma\, x_i^{\delta}}} - 1\right] + \log\alpha\sum_{i=1}^{n}\frac{\gamma x_i^{\gamma}}{1+\gamma x_i^{\delta}}. \quad (6.1)$$

With respect to υ and using Eq. (6.1), we maximize the LLFn as

$$l_\alpha = \frac{n}{\alpha\,\log\alpha} - \frac{n\theta}{\alpha-1} + \gamma(\theta-1)\sum_{i=1}^{n}\frac{x_i^{\delta}}{1+\gamma\, x_i^{\delta}}\left[\alpha - \alpha^{\frac{1}{1+\gamma\, x_i^{\delta}}}\right]^{-1}$$

$$+ \frac{\gamma}{\alpha}\sum_{i=1}^{n}\frac{x_i^{\delta}}{1+\gamma\, x_i^{\delta}}, \quad (6.2)$$

$$l_\theta = \frac{n}{\theta} - n\log(\alpha-1) + \sum_{i=1}^{n}\log\left[\alpha^{\frac{\gamma\, x_i^{\delta}}{1+\gamma\, x_i^{\delta}}} - 1\right], \quad (6.3)$$

$$l_\delta = \frac{n}{\delta} + \sum_{i=1}^{n}\log x_i + \gamma\sum_{i=1}^{n}\frac{x_i^{\delta}\log x_i}{1+\gamma\, x_i^{\delta}}\left[\frac{\log\alpha}{1+\gamma\, x_i^{\delta}} - 2\right]$$

$$+ \gamma(\theta-1)\log\alpha\sum_{i=1}^{n}\left[1 - \alpha^{-\frac{\gamma\, x_i^{\delta}}{1+\gamma\, x_i^{\delta}}}\right]^{-1}\frac{x_i^{\delta}\log x_i}{\left(1+\gamma\, x_i^{\delta}\right)^2}, \quad (6.4)$$

$$l_{\gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}} + (\theta-1)\log\alpha\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{\left(1+\gamma\,\mathrm{x_i^{\delta}}\right)^2}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]^{-1} + \log\alpha\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{\left(1+\gamma\,\mathrm{x_i^{\delta}}\right)^2}. \tag{6.5}$$

Setting $l_{\alpha} = 0, l_{\theta} = 0, l_{\delta} = 0$ and $l_{\gamma} = 0$ of Eqs. (6.2) to (6.5) provides normal equations (NEqs) for MLEs $\hat{\upsilon} = \left(\hat{\alpha},\hat{\theta},\hat{\gamma},\hat{\delta},\right)^{T}$. Analytically, simultaneous solution of these NEqs is tedious. So, we compute numerical solution using iterative method in R language for four parameter EAW model. To build confidence intervals (CIs) for the parameters of the EAW model in asymptotic sense, the required observed information matrix (IMx) $J(\hat{\upsilon})$ is given below due to complications in expected IMx,

$$J(\hat{\upsilon}) = -\begin{pmatrix} l_{\alpha\alpha} & l_{\alpha\theta} & l_{\alpha\gamma} & l_{\alpha\delta} \\ - & l_{\theta\theta} & l_{\theta\gamma} & l_{\theta\delta} \\ - & - & l_{\gamma\gamma} & l_{\gamma\delta} \\ - & - & - & l_{\delta\delta} \end{pmatrix},$$

Where $\upsilon = (\alpha,\theta,\gamma,\delta)^{T}$. The elements of observed IMx in explicit expressions are given below.

$$l_{\alpha\alpha} = -\frac{n(\log\alpha+1)}{(\alpha\log\alpha)^2} + \frac{n\theta}{(\alpha-1)^2} + (1-\theta)\gamma\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}\left[1-\frac{1}{1+\gamma\,\mathrm{x_i^{\delta}}}\alpha^{-\frac{\gamma\,\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]\left[\alpha-\alpha^{\frac{1}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]^{-2} - \frac{\gamma}{\alpha^2}\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}},$$

$$l_{\alpha\theta} = -\frac{n}{\alpha-1} + \gamma\sum_{i=1}^{n}\frac{\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}\left[\alpha-\alpha^{\frac{1}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]^{-1}, \tag{6.7}$$

$$l_{\alpha\delta} = (\theta - 1)\gamma \sum_{i=1}^{n} \frac{\mathrm{x_i^\delta} \log x_i}{1 + \gamma\, \mathrm{x_i^\delta}} \left[\frac{1}{\alpha - \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}}\right] \left\{\frac{1}{1 + \gamma\, \mathrm{x_i^\delta}} - \gamma \log \alpha \frac{\mathrm{x_i^\delta} \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}}{\left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2 \left[\alpha - \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}\right]}\right\} + \frac{\gamma}{\alpha} \sum_{i=1}^{n} \frac{\mathrm{x_i^\delta} \log x_i}{\left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2}, \tag{6.8}$$

$$l_{\alpha\gamma} = (\theta - 1) \sum_{i=1}^{n} \frac{\mathrm{x_i^\delta}}{1 + \gamma\, \mathrm{x_i^\delta}} \left[\frac{1}{\alpha - \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}}\right] \left\{\frac{1}{1 + \gamma\, \mathrm{x_i^\delta}} - \frac{\gamma \log \alpha\, \mathrm{x_i^\delta} \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}}{\left[\alpha - \alpha^{\frac{1}{1+\gamma\, \mathrm{x_i^\delta}}}\right] \left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2}\right\} + \frac{1}{\alpha} \sum_{i=1}^{n} \frac{\mathrm{x_i^\delta}}{\left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2}, \tag{6.9}$$

$$l_{\theta\theta} = -\frac{n}{\theta^2}, \tag{6.10}$$

$$l_{\theta\delta} = \gamma \log \alpha \sum_{i=1}^{n} \left[\frac{\mathrm{x_i^\delta} \log x_i}{1 - \alpha^{-\frac{\gamma\, \mathrm{x_i^\delta}}{1+\gamma\, \mathrm{x_i^\delta}}}}\right] \frac{1}{\left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2}, \tag{6.11}$$

$$l_{\theta\gamma} = \log \alpha \sum_{i=1}^{n} \frac{\mathrm{x_i^\delta}}{\left(1 + \gamma\, \mathrm{x_i^\delta}\right)^2} \left[1 - \alpha^{-\frac{\gamma\, \mathrm{x_i^\delta}}{1+\gamma\, \mathrm{x_i^\delta}}}\right]^{-1}, \tag{6.12}$$

$$l_{\delta\delta} = -\frac{n}{\delta^2} - 2\gamma \sum_{i=1}^{n} \frac{\mathrm{x}_\mathrm{i}^{\delta}[\log x_i]^2}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}$$
$$+\gamma(\theta-1)\log\alpha \sum_{i=1}^{n} \frac{[\log x_i]^2 \mathrm{x}_\mathrm{i}^{\delta}}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}\right]^{-1}\left\{\frac{1-\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}\right.$$
$$\left.-\gamma\log\alpha \frac{\mathrm{x}_\mathrm{i}^{\delta}}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}\right]^{-1}\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}\right\}$$
$$+\gamma\log\alpha \sum_{i=1}^{n} \frac{[\log x_i]^2 \mathrm{x}_\mathrm{i}^{\delta}\left(1-\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^3}, \tag{6.13}$$

$$l_{\delta\gamma} = -2\sum_{i=1}^{n} \frac{\mathrm{x}_\mathrm{i}^{\delta}\log x_i}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}$$
$$+(\theta-1)\log\alpha \sum_{i=1}^{n}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}\right]^{-1} \frac{\mathrm{x}_\mathrm{i}^{\delta}\log x_i}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}\left\{\frac{1-\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}\right.$$
$$\left.-\frac{\gamma\log\alpha\,\mathrm{x}_\mathrm{i}^{\delta}\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^2}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}{1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}}}\right]^{-1}\right\}$$
$$+\log\alpha\sum_{i=1}^{n}\frac{\left(1-\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)\mathrm{x}_\mathrm{i}^{\delta}\log x_i}{\left(1+\gamma\,\mathrm{x}_\mathrm{i}^{\delta}\right)^3}, \tag{6.14}$$

$$l_{\gamma\gamma} = -\frac{n}{\gamma^2} + 2\sum_{i=1}^{n} \frac{\mathrm{x_i^{2\delta}}}{\left(1+\gamma\,\mathrm{x_i^{\delta}}\right)^2}$$

$$+ (1-\theta)\log\alpha \sum_{i=1}^{n} \frac{\mathrm{x_i^{2\delta}}}{\left(1+\gamma\,\mathrm{x_i^{\delta}}\right)^3}\left[1-\alpha^{-\frac{\gamma\,\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]^{-1}\left\{2\right.$$

$$\left.+\log\alpha\left[1-\alpha^{-\frac{\gamma\,\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right]^{-1}\alpha^{-\frac{\gamma\,\mathrm{x_i^{\delta}}}{1+\gamma\,\mathrm{x_i^{\delta}}}}\right\}$$

$$- 2\log\alpha \sum_{i=1}^{n} \frac{\mathrm{x_i^{2\delta}}}{\left(1+\gamma\,\mathrm{x_i^{\delta}}\right)^3}. \tag{6.15}$$

When parameters follow usual regularity conditions and lay within the interior of parameter space but not on boundary then

$$\sqrt{n}(\hat{v}-v) \simeq N_4\left(0, I^{-1}(v)\right). \tag{6.16}$$

In Eq. (6.16), $I(v)$ is the expected IMx. Its behavior is asymptotically valid if $I(v)$ is replaced by $J(\hat{v})$ at $\hat{v}$. With asymptotic multivariate normal distribution (NDn) $N_4\left(0, J^{-1}(\hat{v})\right)$, the large sample $100(1-\emptyset)\%$ CIs of $\hat{v}$ for four parameter EAW model are as follows:

$$\hat{v} \pm Z_{\frac{\emptyset}{2}}\sqrt{J^{-1}(\hat{v})},$$

Where $\emptyset$ is the level of significance and $Z_{\frac{\emptyset}{2}}$ is upper $\frac{\emptyset}{2}$th percentile regarding standard NDn.

7. EMPIRICAL ILLUSTRATION

In empirical illustrations, the analysis is based on parameters estimates and goodnessof fit (GoFt) results of the model. Here, we check the efficiency of the four parameter EAW model by fitting several datasets. We compute the GoFt of the proposed model and make comparisons with other competitive models (CMs) in the same sense. To compare CMs, we obtained measures of GoFt statistics, such as negative LLFn value (-l), Anderson-Darling (A^*), Cramér-von Mises (W^*), and Kolmogrov-Smirnov (KSv) statistic with p-value. Normally, smaller values of GoFt statistics with a high p-value show better fit to the data. For more detail, see [28]. All computations are done in R language.

7.1. Application of the Four Parameter EAW Model on Water Runoff (in mm)

The dataset is taken from [29] as 0.80, 1.60, 1.80, 8.10, 0.80, 0.90, 3.40, 2.50, 216.20, 59.30, 30.70, 78.20, 0.60, 7.30, 17.0, 16.80, 2.0, 24.10, 181.70, 2.0, 6.10, 5.10, 2.0, 146.80, 1.10, 0.80, 13.30, 20.50, 2.90, 6.0, 89.10, 33.50, 7.20, 75.90.

The descriptive statistics for water runoff dataset are n=34, Min.=0.60, Max.=216.20, Mean=31.36, Median=6.65, 1st Qu.=2.00, 3rd Qu.=29.05, Standard deviation=53.75, Skewness=2.22, and Kurtosis=4.07. Figure 11 describes Box and TTT plots. For this data, we compared the fits of the EAW model with the Kumaraswamy exponential (KmE) by [30], alpha power Weibull (APW) by [31], exponentiated exponential (EE) by [32], alpha power exponential (APE) by [21], Weibull (Wbl) by [1], and exponential (Exp) distributions.

The PDFn of the competitive models are as follows:

- The KmE model

$$\phi(x;a,b,\delta)=ab\delta e^{-\delta x}\left(1-e^{-\delta x}\right)^{a-1}\left(1-\left(1-e^{-\delta x}\right)^{a}\right)^{b-1};\ x,a,b,\delta>0.$$

- The APW model

$$\phi(x;\alpha,\gamma,\delta)=\frac{\delta\gamma\log\alpha}{\alpha-1}\alpha^{1-e^{-\delta x^{\gamma}}}e^{-\delta x^{\gamma}}x^{\gamma-1};\ \alpha>0,\alpha\neq1,\delta,\gamma,x>0.$$

- The EE model

$$\phi(x;a,b)=abe^{-bx}(1-e^{-bx})^{a-1};x,a,b>0.$$

- The APE model

$$\phi(x;b,\alpha)=\frac{b\log\alpha}{\alpha-1}\alpha^{1-e^{-bx}}\ e^{-bx};\ \alpha>0,\alpha\neq1,x,b>0.$$

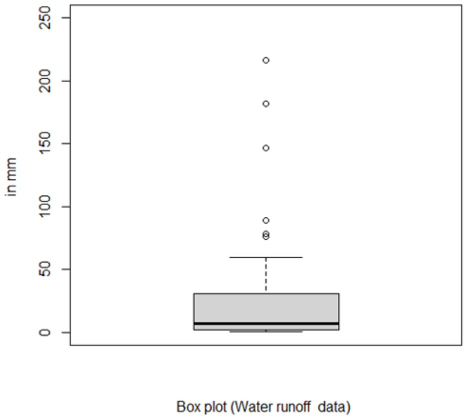
- The Wbl model

$$\phi(x;\gamma,\delta)=\gamma\delta x^{\gamma-1}\mathrm{e}^{-\delta x^{\gamma}};\quad \gamma,\delta,x>0.$$

- The Exp model

$$\phi(x;\alpha)=\alpha\ e^{-\alpha x};x,\alpha>0.$$

Using water runoff data, the MLEs and standard errors of all discussed models are provided in Table 3, while the GoFt statistics results of all fitted models are given in Table 4. From Table 4, we observe that EAW model has the lowest GoFt statistics and the highest *p*-value against all fitted CMs. Furthermore, Figure 12 also support these results



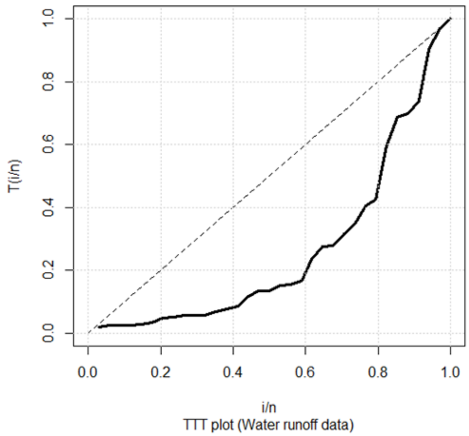


Figure 11. Box Plot and TTT Plot

Table 3. MLEs and Standard Errors of EAW Model Parameters with Other CMs for Water Runoff Data.

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{a}$	$\hat{b}$
EAW	4.08585 (30.96092)	1.89895 (4.51507)	1.10710 (0.01823)	0.65014 (0.00244)	- -	- -
KmE	- -	- -	- -	0.00356 (0.00208)	0.54371 (0.08953)	3.62201 (1.78116)
APW	0.11837 (0.20140)	- -	0.69392 (0.09341)	0.06822 (0.04738)	- -	- -
EE	- -	- -	- -	- -	0.46004 (0.09274)	0.01801 (0.00513)
APE	0.04116 (0.04537)	- -	- -	- -	- -	0.01805 (0.00578)
Wbl	- -	- -	0.59149 (0.07608)	0.17313 (0.05783)	- -	- -
Exp	0.03189 (0.00546)	- -	- -	- -	- -	- -

Table 4. Statistics $-l$, A^*, W^*, KSv, and p-value of the EAW Model with Other CMs for Water Runoff Data.

Models	$-l$	A^*	W^*	KSv	p −value
EAW	138.96810	0.597317	0.070066	0.091398	0.9389
KmE	140.98550	1.075832	0.175093	0.156916	0.3724
APW	139.47030	0.817084	0.123572	0.115470	0.7551
EE	142.26470	1.485958	0.271831	0.189210	0.1752
APE	144.21170	3.670700	0.530737	0.231408	0.0524
Wbl	140.33880	0.970639	0.150244	0.138880	0.5283
Exp	151.14370	9.032758	1.341393	0.360580	0.0003

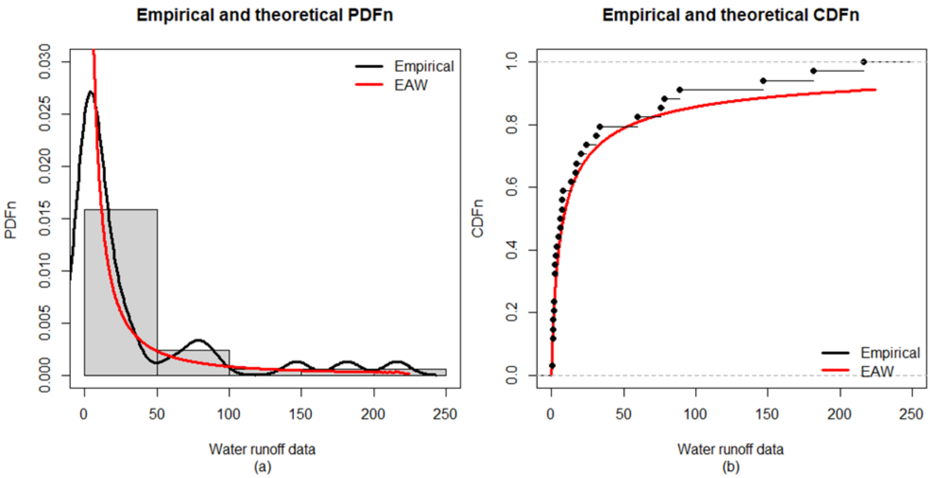


Figure 12. Histogram, Empirical and Theoretical PDFn, and Empirical and Theoretical CDFn of EAW Model for Water Runoff Data

8. CONCLUSION

In this study, a new model of four parameter labeled as 'exponentiated alpha Weibull (EAW)' is developed. It is employed to analyze the distribution of extreme events. By employing EAW model to historical flood data, one can estimate return periods and analyze the potential strength of future floods. It is also useful for designing storm water management system. Moreover, various statistical properties of the EAW model are explored in this study. ML method is applied for parameters estimates and simulation study is conducted to check the behavior of the parameters. GoFt statistics of EAW model are compared with other CMs to identify how well the developed model performs. The current study establishes that the newly developed model yields superior graphical and numerical results.

CONFLICT OF INTEREST

The authors of the manuscript have no financial or non-financial conflict of interest in the subject matter or materials discussed in this manuscript.

DATA AVALIABILITY STATEMENT

Data availability is not applicable as no new data was created.

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No funding has been received for this research.

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