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Semi-Analytical Solutions of Time-Fractional KdV and Modified KdV Equations

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Abstract

In this paper, semi-analytical solutions of time-fractional Korteweg-de Vries (KdV) equations are obtained by using a novel variational technique. The method is based on the coupling of Laplace Transform Method (LTM) with Variational Iteration Method (VIM) and it was implemented on regular and modified KdV equations of fractional order in Caputo sense. Correction functionals were used in general Lagrange multipliers with optimality conditions via variational theory. The implementation of this method to non-linear fractional differential equations is quite easy in comparison with other existing methods.

Keywords: Time-Fractional KdV equations, Variational Iteration Method (VIM), Laplace Variational Method (LVM), non-linear Fractional Differential Equations

Introduction

The majority of phenomena in science and engineering are modeled using non-linear differential equations, especially non-linear partial differential equations such as Korteweg-de Vries equation [1, 2], non-linear Schro¨ndinger equation [3, 4], AC power flow model [5], Richard's equation for unsaturated water flow [6, 7], general relativity [8], etc.

In the recent decades, fractional derivative has been widely used in different non-linear problems to better understand the complex phenomena which not only agree with the current state of the solution but also with its historical background. This fact makes fractional calculus special so that there are various fractional operators and any scientist modelling the real world phenomena can choose the operator that best fits the model [18]. In order to model the real world problems, people have obtained new fractional operators with non-local and non-singular kernels [18].

The Variational Iteration Method (VIM) was first presented by the Chinese mathematician Ji-Huan He [17]. The key property of the method is its flexibility and its ability to solve non-linear differential equations using a very simple procedure. The results are acceptable and have been implemented to an extensive class of non-linear problems [17]. Laplace Transform Method (LTM) is a powerful tool which has been used during the last few decades to solve not only ODEs with constant and variable coefficients but also PDEs.

Recently, active attention has been paid to couple more than one techniques to solve a problem in order to get better results with rapid convergence [16]. In this paper, two techniques VIM and LTM are combined and labelled as Laplace Variational Method (LVM). The key extracts of this semi-analytical technique are (a) to use initial condition (and avoid boundary conditions without any discretization) [16], (b) linearization and (c) restrictive assumptions to solve the non-linear fractional differential equations.

The method description of the proposed technique LVM is presented in section 2. In section 3, it is employed to obtain the solutions of the time-fractional differential equations, representing obscure non-linear phenomena in a very simple way. In section 4, the conclusion is drawn and discussion is made about the effectiveness of the proposed method and the solutions of shallow water waves.

Here, we present the related definitions of Laplace Transform and the fractional derivative of Laplace Transform.

1.1 Laplace Transform

The Laplace Transform of a function $u(t)$ is defined as

$$\mathcal{F}(s) = \mathcal{L}\{\phi(t)\} = \int_0^\infty e^{-st}\,\phi\,(t)dt.$$

1.2 Laplace Transform of Fractional Derivative in Caputo Sense

The Laplace Transform of the fractional derivative $D^{\alpha}\left(f(t)\right)$,

$$\mathcal{L}\{D_t^{\alpha}\phi(t)\} = s^{\alpha}F(s) - \sum_{q=0}^{n-1} s^{\alpha-q-1}\,\phi^{q}(0),$$
$$n-1 < n\alpha \le n. \tag{1.1}$$

2. Method Description

To understand the method description of LVM, first consider the generalized fractional differential equation having the following form,

$$D_t^{\alpha}\phi(\xi,t) + L\big(\phi(\xi,t)\big) + N\big(\phi(\xi,t)\big) = f(\xi,t); \tag{2.1}$$
$$\phi(\xi,0) = \phi_0;\ \ n-1 < n\alpha \le n;\quad t > 0$$

Where D_t^{α} is time fractional derivative in Caputo sense, L is linear operator, N is non-linear operator and $f(\xi,t)$ is the known function. The recursive relation after applying the Laplace Transform on Eq. (2.1) is as follows,

$$\phi_{n+1}(\xi,s) = \phi_n(\xi,s) + \lambda\mathcal{L}\left\{D_t^{\alpha}\phi(\xi,t) + L\big(\phi_n(\xi,t)\big) + N\left(\phi_n(\xi,t)\right) - f(\xi,t)\right\}, \tag{2.2}$$

Where λ is the general Lagrange multiplier. Here, taking the variation in order to obtain the value of Lagrange multiplier using optimality conditions we obtain

$$\frac{\delta\phi_{n+1}(\xi,t)}{\delta\phi_n(\xi,t)}) = 0, \quad \text{and} \quad \delta\widetilde{\phi_n} = 0,$$

That gives $\lambda = \frac{-1}{s^{\alpha}}$. On substituting the value of λ and taking Inverse Laplace Transform of Eq. (2.2), we obtain

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\,\mathcal{L}\left\{D_t^{\alpha}\phi_n(\xi,t) + L\big(\phi_n(\xi,t)\big) + N\big(\phi_n(\xi,t)\big) - f(\xi,t)\right\}\right\}. \tag{2.3}$$

On substituting $n = 0,1,2,...,$ we obtain the successive iterations$\phi 1, \phi 2, \phi 3$,... In the next section, we implement the proposed LVM to solve non-linear fractional differential equations.

3. Applications

This section is dedicated to examples on which LVM has been applied.

More specifically, we apply the method to solve different non-linear time-fractional KdV equations.

3.1 Example 1

Here, we solve time-fractional regular KdV equations using LVM.

$$D_t^{\alpha}\phi + \alpha_1\phi\phi_{\xi} + \beta_1\phi_{\xi\xi\xi} = 0; \quad 0 < \alpha \leq 1, \tag{3.1}$$

having the initial condition

$$\phi(\xi, 0) = \phi_0 = \text{sech}^2\beta_1\xi,$$

where $\alpha_1 = \frac{c_0}{2K^2}(\in c\lambda_3)$ is the non − liner, $\beta_1 = \frac{c_0 h^2}{6}$ represents the dispersion parameter.

Now, we apply VIM on Eq. (3.1). We get

$$\phi_{n+1}(\xi, \mathrm{t}) = \phi_n(\xi, \mathrm{t}) + \lambda\left\{D_t^{\alpha}\phi_n(\xi, \mathrm{t}) + \alpha_1\phi_n\frac{\partial\phi_n}{\partial\xi} + \frac{\partial^3\phi_n}{\partial\xi^3}\right\}. \tag{3.2}$$

Applying Laplace Transform on Eq. (3.2), we obtain

$$\phi_{n+1}(\xi, s) = \phi_n(\xi, \mathrm{s}) + \mathcal{L}\left\{\lambda\left\{D_t^{\alpha}\phi_n(\xi, s) + \alpha_1\phi_n\frac{\partial\phi_n}{\partial\xi} + \frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}. \tag{3.3}$$

Now, using the definition of Laplace Transform of fractional derivative given in Eq. (3.4) in Eq. (3.3)

$$\mathcal{L}\{D_t^{\alpha}\phi_n(\xi, \mathrm{t})\} = s^{\alpha}\phi_n(\xi, s) - s^{\alpha-1}\phi_n(\xi, 0), \tag{3.4}$$

we get

$$\phi_{n+1}(\xi, s) = \phi_n(\xi, s) + \lambda\{s^{\alpha}\phi_n(\xi, s) - s^{\alpha-1}\phi_n(\xi, 0)\} + \lambda\mathcal{L}\left\{\alpha_1\widetilde{\phi_n}\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}.$$

One can obtain the value of Lagrange multiplier using the optimality conditions, that is, $\lambda = \frac{-1}{s^{\alpha}}$. On substituting the value of λ, we have

$$\phi_{n+1}(\xi,s) = \phi_n(\xi,s) - \frac{1}{s^{\alpha}}\{s^{\alpha}\phi_n(\xi,s) - s^{\alpha-1}\phi_n(\xi,0)\} - \frac{-1}{s^{\alpha}}\mathcal{L}\left\{\alpha_1\widetilde{\phi_n}\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}.$$

Taking Inverse Laplace Transform and simplifying it, we obtain

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_n(\xi,t) + \alpha_1\phi_n\frac{\partial\phi_n}{\partial\xi} + \beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}.$$

For n = 0,1,2,..., we have the successive approximations $\phi_{1,}\phi_2,\phi_3$ ...

$$\phi_1(\xi,t) = \phi_0(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_0 + \alpha_1\phi_0\frac{\partial\phi_0}{\partial\xi} + \beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}.$$

Putting the values of $\phi_1(\xi,t), \phi_1\frac{\partial\phi_0}{\partial\xi}$ and $\frac{\partial^3\phi_n}{\partial\xi^3}$, we find

$$\phi_1(\xi,t) = a\ \text{seach}^2\beta_1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\{-2a^2\alpha_1\beta_1\text{seach}^2\beta_1\ \text{than}\beta_1\xi + \beta_1(16a\beta_1^3\text{seach}^4\beta_1\text{than}\beta_1\xi - 8ab_1^3\text{seach}^2\ \beta_1\xi\tanh^3\beta_1\xi\}.$$

ϕ_1 can be obtained in the same manner.

The solution $\phi\ (\xi,t)$ *can be obtained by taking sun of* $\phi 0, \phi 1,$ $\phi 2,\ldots, i.e.$,

$$\phi(\xi,t) = \Sigma(\phi_0 + \phi_1 + \phi_2 + \ldots).$$

3.2 Example 2

In this example, we consider another time-fractional KdV equation of the form

$$D_t^{\alpha}\phi + \beta_1\phi_{\xi\xi\xi} = 0; \quad 0 < \alpha \leq 1,$$

subject to the initial condition

$$\phi(\xi,t) = \phi_0 = \frac{a}{\cosh^2\beta_1\xi},$$

where $\beta_1 = \frac{c_0h^2}{6}$ represents dispersion. Applying VIM, we find

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) + \lambda\left\{D_t^{\alpha}\phi_n(\xi,t) + \beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}. \qquad (3.5)$$

Applying Laplace Transform of Eq. (3.5), we have

$$\phi_{n+1}(\xi,s) = \phi_n(\xi,s) + \lambda\mathcal{L}\left\{\left\{D_t^{\alpha}\phi_n(\xi,t) + \beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}. \qquad (3.6)$$

Using the definition Eq. (1.1), we get the expression

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,s) + \lambda\{s^{\alpha}\phi_n(\xi,s) - s^{\alpha-1}\phi_n(\xi,0)\} + \lambda\mathcal{L}\left\{\beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}. \qquad (3.7)$$

One can find the value of general Lagrange multiplier $\lambda = \frac{-1}{s^{\alpha}}$ using the optimality conditions $\frac{\delta\phi_n+1(\xi,s)}{\delta\phi_n(\xi,s)} = 0$ and $\delta\widetilde{\phi_n} = 0$.

By putting the value of λ and applying Inverse Laplace Transform on Eq. (3.7), we get

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_n + \beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}. \qquad (3.8)$$

The following iterations $\phi_{1,}\phi_{2,}\phi_3 \ldots$ can be obtained from the Eq. (3.8),

$$\phi_1(\xi,t) = \phi_0(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_0 + \beta_1\frac{\partial^3\phi_0}{\partial\xi^3}\right\}\right\}$$

$$\phi_2(\xi,t) = \phi_1(\xi,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_1 + \beta_1\frac{\partial^3\phi_1}{\partial\xi^3}\right\}\right\}$$

Further, using the values of $\phi_0(\xi,t)\, and\, \frac{\partial^3\phi_0}{\partial\xi^3}$ the above expressions will end up with

$$\phi_1(\xi,t) = a\, sech^2\beta_1\xi - \frac{t^\alpha}{\Gamma(1+\alpha)}\{\beta_1(16a\beta_1^3 sech^4\beta_1\xi tanh\beta_1\xi - 8a\beta_1^3 sech^2\beta_1\xi tanh^3\beta_1\xi)\}$$

$$\phi_1(\xi,t) = a\, sech^2\beta_1\xi - \frac{1}{\Gamma(1+\alpha)}\{t^\alpha\beta_1(16a\beta_1^3 sech^4\beta_1\xi tanh\beta_1\xi - 8a\beta_1^3 sech^2\beta_1\xi tanh^3\beta_1\xi)\} + \frac{1}{\big(\Gamma(2\alpha)\big)\big(\Gamma(1+\alpha)\big)(\Gamma(1+2\alpha))}\Big\{2at^{(-1+\alpha)}\beta_1^4 sech^5\beta_1\xi(-t^\alpha\alpha\Gamma\alpha\Gamma(1+2\alpha)\,(-11sinh\beta_1\xi) + sinh3\beta_1\xi)\big(t^\alpha\beta_1^4(-1208+1191cosh2\beta_1\xi - 120cosh4\beta_1\xi + cosh6\beta_1\xi)\Gamma(1+\alpha)sech^3\beta_1\xi + \Gamma(1+2\alpha)(-11sinh\beta_1\xi + sinh3\beta_1\xi)\big)\Big\}$$

The solution $\phi(\xi,t)$ can be obtained by taking the sum of $\phi 0, \phi 1,$

$\phi 2, \ldots, i.e.,$

$$\phi(\xi,t) = \Sigma(\phi_0 + \phi_1 + \phi_2 + \ \ldots..)$$

3.3 Example 3

Here, we consider the time − fractional KdV − like equation, i. e.,

$$D_t^\alpha\phi + c_0\phi_\xi + \alpha_1\big(\phi_\xi^2 - \phi\phi_\xi\big) + \beta_1\phi_{\xi\xi\xi} = 0; \quad 0 < \alpha \le 1,$$

subject to the initial condition

$$\phi(\xi,0) = \phi_0 = \frac{a}{\cosh^2\beta_1\xi},$$

where $\alpha_1 = \frac{c_0}{2k^2}(\in c\lambda_3)$ represents non − linear and $\beta_1 = \frac{c_0h^2}{6}$ represents the dispersion parameter.

Now, implementing VIM, we have

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) + \lambda\left\{D_t^{\alpha}\phi_n + c_0\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \alpha_1\left(\left(\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right)^2 - \widetilde{\phi_n}\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right) + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}. \quad (3.9)$$

Applying Laplace Transform on Eq. (3.9), we obtain

$$\phi_{n+1}(\xi,s) = \phi_n(\xi,s) + \mathcal{L}\left\{\lambda\left\{D_t^{\alpha}\phi_n + c_0\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \alpha_1\left(\left(\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right)^2 - \widetilde{\phi_n}\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right) + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}\right\}.$$

Using the definition Eq. (1.1), we have

$$\phi_{n+1}(\xi,s) = \phi_n(\xi,s) + \lambda\{s^{\alpha}\phi_n(\xi,s) - s^{\alpha-1}\phi_n(\xi,0)\} + \lambda\mathcal{L}\left\{c_0\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \alpha_1\left(\left(\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right)^2 - \widetilde{\phi_n}\frac{\partial\widetilde{\phi_n}}{\partial\xi}\right) + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}. \quad (3.10)$$

The value of Lagrange multiplier $\lambda = \frac{-1}{s^{\alpha}}$ sα can be obtained using the optimality conditions, i.e.,

$$\frac{\delta\phi_{n+1}(\xi,s)}{\delta\phi_n(\xi,s)} = 0, \text{ and } \delta\widetilde{\phi_n} = 0.$$

On substituting the Lagrange multiplier and applying inverse Laplace Transform on Eq. (3.10), we get

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) - \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left\{D_t^{\alpha}\phi_n + c_0\frac{\partial\phi_n}{\partial\xi} + \alpha_1\left(\left(\frac{\partial\phi_n}{\partial\xi}\right)^2 - \phi_n\frac{\partial\phi_n}{\partial\xi}\right) + \beta_1\frac{\partial\phi_n}{\partial\xi^3}\right\}\right\}.$$

For n = 0,1,2,..., we get the approximations ϕ_1, ϕ_2, ϕ_3 ... that are

$$\phi_1(\xi,t) = \phi_0(\xi,t) - \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left\{D_t^{\alpha}\phi_0 + c_0\frac{\partial\phi_0}{\partial\xi} + \alpha_1((\frac{\partial\phi_0}{\partial\xi})^2 - \phi_0\frac{\partial\phi_n}{\partial\xi}) + \beta_1\frac{\partial\phi_0}{\partial\xi^3}\right\}\right\}. \quad (3.11)$$

Now, putting the values of$\phi_0(\xi,t), \frac{\partial\phi_0}{\partial\xi})^2, \phi_0\frac{\partial\phi_0}{\partial\xi}$ and $\frac{\partial^3\phi_0}{\partial\xi_3}$ Eq. (3.11) and simplifying, we obtain

$$\phi_1(\xi,t) = a\,sech^2\beta_1\xi - \frac{t^{\propto}}{\Gamma(1+\propto}\{-2ac_0\beta_1\mathrm{sech}^2\beta_1\xi\tanh\beta_1\xi + \alpha_1(2a^2\beta_1\mathrm{sech}^4\beta_1\xi\tanh\beta_1\xi + 4a^2\beta_1^2\mathrm{sech}^4\beta_1\xi\tanh^2\beta_1\xi) + \beta_1(16a\beta_1^3\mathrm{sech}^4\beta_1\xi\tanh\beta_1\xi - 8a\beta_1^3\mathrm{sech}^2\beta_1\xi\tanh^3\beta_1\xi)\}.$$

ϕ_2 can be evaluated in the same manner and $\phi(\xi,t)$ can be obtained by taking the sum of

ϕ_0, ϕ_1, ϕ_2, i.e.,

$$\phi(\xi,t) = \Sigma(\phi_0 + \phi_1 + \phi_2 + \ldots\ldots).$$

3.4 Example 4: Gardner's Equation

Here, we solve time-fractional Gardner's equation using Variational Iteration Transform Method,

$$D_t^{\alpha}\phi + \alpha_1\phi(1+\phi)\phi_{\xi} + \beta_1\phi_{\xi\xi\xi} = 0,$$

along with the condition

$$\phi(\xi,0) = \phi_0 = \frac{a}{\cosh^2\beta_1\xi},$$

where $\alpha_1 = \frac{c_0}{2k^2}(\in c\lambda_3)$ represents the non-linear and $\beta_1 = \frac{c_{0h^2}}{6}$ represents the dispersion parameter. Implementing VIM,

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) + \lambda\left\{D_t^{\alpha}\phi_n + \alpha_1\widetilde{\phi_n}(1+\widetilde{\phi_n})\frac{\partial\widetilde{\phi_n}}{\partial\xi} + \beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}.$$

Now, applying Laplace Transform that gives

$$\phi_{n+1}(\xi,s)=\phi_n(\xi,s)+\ \mathcal{L}\ \left\{\lambda\left\{D_t^{\alpha}\phi_n+\alpha_1\widetilde{\phi_n}(1+\widetilde{\phi_n})\frac{\partial\widetilde{\phi_n}}{\partial\xi}+\beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}\right\}.$$

Using the definition of Laplace Transform of fractional derivative (Eq. 1.1) in the above equation, we have

$$\phi_{n+1}(\xi,s)=\phi_n(\xi,s)+\lambda\left\{s^{\alpha}\phi_n(\xi,s)-s^{\alpha-1}\phi_n(\xi,0)\right\}+\mathcal{L}\left\{\lambda\left\{\alpha_1\widetilde{\phi_n}(1+\widetilde{\phi_n})\frac{\partial\widetilde{\phi_n}}{\partial\xi}+\beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}\right\},$$

While general Lagrange multiplier $\lambda=\frac{-1}{s^{\alpha}}$ can be evaluated using the optimality conditions

$$\frac{\delta\phi_{n+1}(\xi,s)}{\delta v_n(\xi,s)}=0,\ \text{ and }\ \delta\widetilde{\phi_n}=0,$$

$$\phi_{n+1}(\xi,s)=\phi_n(\xi,s)-s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_n+\alpha_1\widetilde{\phi_n}\left(1+\widetilde{\phi_n}\right)\frac{\partial\widetilde{\phi_n}}{\partial\xi}+\beta_1\frac{\partial^3\widetilde{\phi_n}}{\partial\xi^3}\right\}.$$

Here, we apply Inverse Laplace Transform that gives

$$\phi_{n+1}(\xi,t)=\phi_n(\xi,t)-\mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{D_t^{\alpha}\phi_n+\alpha_1\widetilde{\phi_n}(1+\phi_n)\frac{\partial\phi_n}{\partial\xi}+\beta_1\frac{\partial^3\phi_n}{\partial\xi^3}\right\}\right\}.$$

For $n=0,1,2,\ldots$, we have the successive approximations ... that are ϕ_0,ϕ_1,ϕ_2

$$\phi_1(\xi,t)=\phi_0(\xi,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left\{D_t^{\alpha}\phi_0+\alpha_1\phi_0(1+\phi_0)\frac{\partial\phi_0}{\partial\xi}+\beta_1\frac{\partial^3\phi_0}{\partial\xi^3}\right\}\right\}. \tag{3.12}$$

Substituting the values of $\phi_0(\xi,t)$, $\frac{\partial\phi_0}{\partial\xi}$ in Eq. (3.12), we have

$$\phi_1(\xi,t) = a\, sech^2\beta_1\xi - \frac{t^\alpha}{\Gamma(1+\alpha)}\{\alpha_1(-2a^2\beta_1\text{sech}^4\beta_1\xi\tanh\beta_1\xi - 2a^3\beta_1\,\text{sech}^6\beta_1\xi\tanh\beta_1\xi) + \beta_1(16a\,\beta_1^3\text{sech}^4\,\beta_1\xi\tanh\beta_1\xi - 8a\,\beta_1^3\text{sech}^2\,\beta_1\xi\tanh^3\beta_1\xi)\},$$

$\phi_2$can be obtained in a similar way. The solution $\phi(\xi,t)$ is the sum of ϕ_0,ϕ_1,ϕ_2,..., i.e.,

$$\phi(\xi,t) = \Sigma(\phi_0+\phi_1+\phi_2+\ \ldots\ldots).$$

4. Concluding Remarks

The proposed method LVM is understandable with only the basic knowledge of advance calculus; indeed, it is understandable even for the reader who has no background of calculus of variations. It is simple and easy to apply as compared to the more traditional VIM for fractional differential equations. The advantage of this extended variational method is that Laplace Transform helps to expedite the computational cost and can easily be applied to non-linear dynamical systems using user friendly softwares such as Mathematica^{TM} and Maple^{TM}.

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