Scientific Inquiry and Review (SIR)
Volume 9 Issue 3, 2025
$ISSN_{(P)}$: 2521-2427, $ISSN_{(E)}$: 2521-2435
Homepage: https://journals.umt.edu.pk/index.php/SIR

Title:	**Convergence and Ulam-Hyers-Rassias Stability Analyses of Numerical Solutions of Bratu Type Equations using Picard Method**
Author (s):	Saif Ullah, Muzaher Ali, Sana Bajwa, and Ahsan Bilal
Affiliation (s):	Government College University, Lahore, Pakistan
Academic Editor:	Abaid ur Rehman
DOI:	https://doi.org/10.32350/sir.93.01
History:	Received: May 11, 2025, Revised: July 07, 2025, Accepted: July 28, 2025, Published: August 18, 2025
Citation:	Ullah S, Ali M, Bajwa S, Bilal A. Convergence and Ulam-Hyers-Rassias stability analyses of numerical solutions of Bratu type equations using Picard method. *Sci Inq Rev*. 2025;9(3):01–27. https://doi.org/10.32350/sir.93.01
Copyright:	© The Authors
Licensing:	This article is open access and is distributed under the terms of Creative Commons Attribution 4.0 International License
Conflict of Interest:	Author(s) declared no conflict of interest

A publication of
The School of Science
University of Management and Technology, Lahore, Pakistan

Convergence and Ulam-Hyers-Rassias Stability Analyses of Numerical Solutions of Bratu Type Equations using Picard Method

Saif Ullah*, Muzaher Ali, Sana Bajwa, and Ahsan Bilal

Department of Mathematics, Government College University Lahore, Pakistan

ABSTRACT

The Bratu equation is a basic nonlinear boundary value problem with important applications to fuel ignition, thermal combustion, and nanotechnology. The current article presents a new application of the Picard iterative technique to find accurate approximate solutions for this equation. Firstly, the conditions for existence and uniqueness of the solutions are determined. Furthermore, the article provides an explicit formulation of Picard's scheme for second-order ordinary differential equations and its particular implementation to the Bratu type problem. The iterative solutions obtained are analyzed thoroughly for convergence and are proved to be Ulam-Hyers-Rassias stable, a very strong type of stability not yet known for these kind of solutions. Numerical tests for three cases ($\mu = 1$, 2, and the critical value $\mu = 3.51383$) are shown to exemplify outstanding accuracy of the proposed approach. A thorough comparison with known techniques including the Adomian Decomposition Method, Homotopy Perturbation Method, and Variational Iteration Method indicates that the Picard iterative scheme is much better in terms of accuracy since its maximum absolute errors are much smaller.

Keywords: Bratu type equations, convergence analysis, error estimation, existence and uniqueness of solutions, Picard method, Ulam-Hyers-Rassias stability.

Highlights

- The foremost aim of the current article is to compute approximate solutions of BTEs using Picard's method which are new and not described before in the literature.
- The convergence of the sequence $U_n(x)$, generated by Picard's iteration is assessed using the uniform convergence criterion on the interval $x \in [0, 1]$.

*Corresponding Author: dr.saifullah@gcu.edu.pk

- This demonstrates that both exact and Picard's solutions make an excellent agreement.

1. INTRODUCTION

Mathematical modeling of many physical phenomena is achieved in the form of integral equations, differential equations (DEs), or integro-differential equations [1-3]. Bratu type DEs have many applications in practical life. These include fuel ignition, Chandrasekhar model, nanotechnology, thermal reaction, heat transfer by conduction, radiative heat transfer, and chemical reactor theory [4-7]. The scientists and researchers are very keen in the study of Bratu type equations (BTEs). This is because the solutions of such type of equations are more natural and significant due to their important usage in engineering and scientific experimentations. The following equation is called one dimensional Bratu equation [8, 9]

$$U''(x) + \mu \ e^{U(x)} = 0 \quad \text{with} \ \ U(0) = 0, \ U(1) = 0 \ ; \quad \forall \quad x \ \in [\ 0, 1] \qquad (1)$$

where $\mu > 0$ is a physical parameter. Exact solution of above equation (1) is [10]

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\ \ \frac{\vartheta}{2}\right)}{\cosh\ \ \left(\frac{\vartheta}{4}\right)}\right),$$

where ϑ can be obtained from

$$\vartheta = \sqrt{2\ \ \mu}\ \ \cosh\left(\frac{\vartheta}{4}\right).$$

Several analytical, semi-analytical, and numerical schemes have been used to solve BTEs. Decomposition Method (DM) [11], Homotopy Perturbation Method (HPM) [12, 13], Perturbation Iteration Algorithm (PIA) [14], Optimal Homotopy Asymptotic Method (OHAM) [15], Laplace Method (LM) [16], Variational Iteration Method (VIM) [17], Perturbation Method (PM) [18], and B Spline Method (BSM) [19] have been implemented to handle BTEs.

While the above-mentioned approaches have been used successfully, they require intricate calculations, for instance, the formation of Adomian polynomials or determining auxiliary parameters and functions in homotopy-based approaches. The Picard iterative scheme is an attractive alternative considering its simplicity of concept, ease of implementation, and well-established rapid convergence for many nonlinear problems [20-24]. However, its use on the Bratu equation, especially with an extensive study on the stability of its solutions, is an uncharted territory. Therefore, the current article aimed to fill that gap. The novelty of this research is two-pronged: (i) systematic use and numerical verification of the Picard method for BTEs and (ii) pioneering Ulam-Hyers-Rassias stability study of the approximate solutions developed by the iterative scheme.

The study of Ulam-Hyers type stability of various dynamical systems has received considerable attention. The famous Ulam-Hyers stability instigated in a seminar at the Wisconsin University in 1940, where Ulam [25] suggested the problem of the stability for functional equations. A significant development occurred in 1941, when Hyers [26] gave a partial solution to Ulam's problem. After an in-depth analysis of the Ulam-Hyers stability structure, some researchers extended and used the concept of Ulam-Hyers stability, such as generalized Ulam-Hyers stability [27, 28], Ulam-Hyers-Rassias stability [29], and generalized Ulam-Hyers-Rassias stability. Huang et al. [30] analyzed Hyers-Ulam stability, while Qarawani [31] investigated Hyers-Ulam-Rassias stability of nonlinear DEs. Wang et al. considered Ulam-Hyers stability of fuzzy fractional DEs with delay [32, 33].

The foremost aim of the current article is to compute approximate solutions of BTEs using Picard's method which are new and not described before in the literature. The article is organized as: Section 2 covers conditions for existence and uniqueness of solutions, while section 3 comprehends the illustration of Picard method. Section 4 presents the solutions of numerical examples. The convergence and stability analyses of the derived solutions have been performed in section 5 and 6, respectively. Estimation of error and comparison of results are incorporated in section 7, while conclusion of the article is drafted in section 8.

2. CONDITIONS FOR EXISTENCE AND UNIQUENESS OF SOLUTIONS OF BRATU EQUATIONS

The existence and uniqueness of solutions for the Bratu problem are well-established and depend critically on the parameter μ. The foundational result is summarized in the following theorem:

Theorem 1. There are three cases for μ given in equation (1)

- If $\mu = \mu_0$, then there exists one solution,
- If $\mu < \mu_0$, then there exist two solutions,
- If $\mu > \mu_0$, then no solution exists.

Here μ_0 is called the critical value [12, 13] given as:

$$1 = \frac{1}{4}\sqrt{2\ \mu_0}\ \sinh\left(\frac{\vartheta}{4}\right),$$

and its numerical value is

$$\mu_0 = 3.513830719.$$

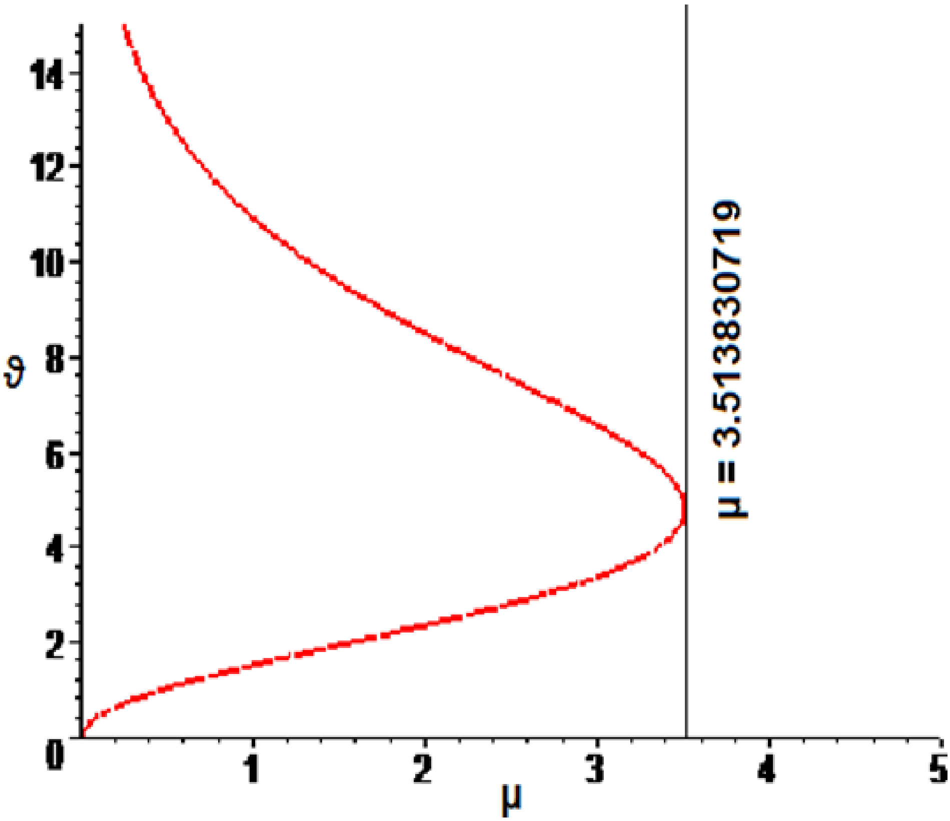


Figure 1. Comparison of μ and ϑ.

Figure 1 shows that when $\mu < 3.513830719$, then ϑ has two values for each value of μ. Hence, there are two solutions for each

$\mu < 3.513830719$. For $\mu = 3.513830719$, only one value of

$\vartheta = 4.798645359$ exists and there is a unique solution. When

$\mu > 3.513830719$, there is no resultant value of ϑ, and in this case, no solution of Bratu equation exists [12, 13].

3. DESCRIPTION OF PICARD METHOD

This section discusses the description of Picard method for second-order ordinary differential equation (ODE) and its implementation to BTE.

3.1. Formulation of Picard Method for Second-Order ODE

Let us take a second-order ODE

$$\mathcal{L}(U(x)) = \mathcal{N}(U(x)) \ , \tag{2}$$

where $\mathcal{L}(U(x))$ is linear operator, while $\mathcal{N}(U(x))$ is nonlinear operator [20]. Writing $\mathcal{L}(U(x))$ in terms of derivative of second-order as

$$\mathcal{L}(U(x)) = U''(x) \ ,$$

while $\mathcal{N}(U(x))$ in terms of $U(x)$ and $U\ '(x)$ as

$$\mathcal{N}(U(x) = (U(x) \ , \ U\ '(x))$$

then, equation (2) yields

$$U''(x) = \mathcal{N}(U(x) \ , U\ '(x)) \ , \tag{3}$$

where

$$U(0) = \alpha \quad \text{and} \quad U\ '(0) = \beta.$$

Solving equation (3), we have

$$U(x) = \alpha + \beta \ \ x + \int_0^x \int_0^x \mathcal{N}(U(x) \ , U\ '(x))dx \ \ dx \ . \tag{4}$$

Hence, the iterative formula is

$$U_{n+1}(x) = \alpha + \beta \ \ x + \int_0^x \int_0^x \mathcal{N}(U_n(x) \ , \ U_n'(x))dx \ \ dx \ , \tag{5}$$

that is Picard's iterative scheme for second-order ODE [21, 22].

3.2. Implementation to Bratu Equation

Let us take equation (1) as

$$U''(x) + \mu \; e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0 \;,\; U(1) = 0 \;;\; \forall \; x \in [\;0\;,1]. \tag{6}$$

Here $\mathcal{L}(U(x)) = U''(x)$ and $\mathcal{N}(U(x)) = \mu \; e^{U(x)}$.

From the above equation (6), we have [12, 34]

$$U''(x) = -\mu \; e^{U(x)}. \tag{7}$$

Implementation of Picard method yields

$$U_{n+1}(x) = \alpha + \beta \; x - \mu \int_0^x \int_0^x e^{U_n(x)} dx \; dx \;, \tag{8}$$

where $U(0) = \alpha = 0$, and β changes as μ changes.

$$\Rightarrow \qquad U_{n+1}(x) = \beta \; x - \mu \int_0^x \int_0^x e^{U_n(x)} dx \; dx \;. \tag{9}$$

This is Picard's iterative scheme for Bratu equation.

4. SOLUTIONS OF BTEs USING PICARD METHOD

In this section, Picard's iterative scheme is used to compute the approximate solutions of various BTEs.

4.1. Example-1

Consider

$$U''(x) + e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0 \;,\; U(1) = 0 \;; \quad \forall \quad x \in [\;0\;,1]. \tag{10}$$

Here, $\mu = 1$. As previously deliberated that for $\mu < 3.513830719$, there exist two solutions of Bratu equation. Thus, equation (10) has two solutions and for finding those solutions

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\frac{\vartheta}{2}\right)}{\cosh\left(\frac{\vartheta}{4}\right)}\right),$$

where ϑ can be obtained from

$\vartheta = \sqrt{2\ \ \mu}\ \ \cosh\left(\frac{\vartheta}{4}\right).$

Since $\mu = 1$, so

$$\vartheta = \sqrt{2}\ \ \cosh\left(\frac{\vartheta}{4}\right). \tag{11}$$

After solving equation (11), we have

$\vartheta_1 = 1.517164598\ \ ,\qquad\qquad \vartheta_2 = 10.9387028.$

Thus, two solutions are obtained, that is, ϑ_1 (called lower solution) and ϑ_2 (called higher solution). However, in the present case, merely lower solution will be used, then

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\frac{1.517164598}{2}\right)}{\cosh\left(\frac{1.517164598}{4}\right)}\right)\ ,$$

$$\Rightarrow U(x) = -2\ \ \ln(0.9321424738\ \ \cosh(0.7585822990\ \ x - 0.3792911495))$$

which is the exact solution of equation (10). Taking $U_0(x) = 0$ and applying Picard method, we get

$$U_1(x) = 0.5493527280\ \ x - 0.5\ \ x^2\ \ ,$$
$$U_2(x) = 0.0003720238095\ \ x^8 - 0.001634978357\ \ x^7 - 0.001651763169\ \ x^6 + 0.01235224910\ \ x^5 + 0.02909214918\ \ x^4 - 0.09155878800\ \ x^3 - 0.5\ \ x^2 + 0.5493527280\ \ x,$$

and

$$\begin{aligned} U_3(x) = {} & 0.5493527280\ x - 0.5\ x^2 - 0.09155878800\ x^3 + 0.02909214918\ x^4 + 0.0183117576\ x^5 - 0.003459802640\ x^6 - 0.001764607014\ x^7 + 0.9322482480 \times 10^{-4} x^8 + 1.580853532 \times 10^{-4} x^9 + 0.4533942348 \times 10^{-5} x^{10} - 1.393134891 \times 10^{-5} x^{11} + 0.06120893625 \times 10^{-6} x^{12} + 0.06540389326 \times 10^{-5} x^{13} + + 0.04400289 \times 10^{-6} x^{14} - 0.03474251 \times 10^{-6} x^{15} - 0.03008674 \times 10^{-7} x^{16} + 0.02236216459 \times 10^{-7}\ x^{17} - 0.02261465929 \times 10^{-8}\ x^{18} \end{aligned} \tag{13}$$

that is the approximate solution of equation (10).

4.2. Example-2

Consider

$U''(x) + 2\ e^{U(x)} = 0$ with $U(0) = 0\ ,\ U(1) = 0\ ;\ \forall\ x \in [\ 0\ , 1].$ (14)

Here, $\mu = 2$. As previously deliberated that for

$\mu < 3.513830719$, there exist two solutions of Bratu equation. Thus, equation (14) has two solutions and for finding those solutions

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\frac{\vartheta}{2}\right)}{\cosh\left(\frac{\vartheta}{4}\right)}\right),$$

where ϑ can be derived from

$$\vartheta = \sqrt{2\ \mu}\ \cosh\left(\frac{\vartheta}{4}\right).$$

Since $\mu = 2$, so

$$\vartheta = \sqrt{2 \times 2}\ \cosh\left(\frac{\vartheta}{4}\right). \quad (15)$$

After solving equation (15), we get

$\vartheta_1 = 2.357551054$, $\vartheta_2 = 8.5071995.$

Thus, two solutions are obtained, that is, ϑ_1 (called lower solution) and ϑ_2 (called higher solution). However, in the present case, merely lower solution will be used, then

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\frac{2.357551054}{2}\right)}{\cosh\left(\frac{2.357551054}{4}\right)}\right),$$

$$\Rightarrow \quad U(x) = -2\ \ln(0.8483379380\ \cosh(1.178775527\ x - 0.5893877635)),$$

which is the exact solution of equation (14). Taking $U_0(x) = 0$ and applying Picard method, we get

$$U_1(x) = 1.248217518 \ x - x^2 \ ,$$
$$U_2(x) = 0.005952380952 \ x^8 - 0.02971946472 \ x^7 + 0.01860156574 \ x^6 + 0.0924087264 \ x^5 + 0.0368294189 \ x^4 - 0.416072506 \ x^3 - x^2 + 1.248217518 \ x, \quad (16)$$

and

$$U_3(x) = -0.115787055 \times 10^{-6} \ x^{18} + 0.130074688 \times 10^{-5} \ x^{17} - 0.460289 \times 10^{-5} \ x^{16} - 0.02644413 \times 10^{-6} \ x^{15} + 0.2586929702 \times 10^{-4} \ x^{14} + 0.2374648306 \times 10^{-4} \ x^{13} - 0.1722400368 \times 10^{-3} \ x^{12} - 0.5966016460 \times 10^{-3} \ x^{11} + 0.001944801524 \ x^{10} + 0.003173141654 x^9 - 0.00655987510 \ x^8 - 0.02640249326 \ x^7 - 0.001165361882 \ x^6 + 0.1664290024 \ x^5 + 0.03682941898 \ x^4 - 0.4160725060 \ x^3 - \ x^2 + 1.248217518 \ x \ , \quad (17)$$

that is the approximate solution of equation (14).

4.3. Example-3

Consider

$$U''(x) + 3.513830719 \ e^{U(x)} = 0$$

with $U(0) = 0, \ U(1) = 0; \ \forall \quad x \in [\ 0 \ , 1].$ (18)

Here, $\mu = 3.513830719$. As previously deliberated that for $\mu = 3.513830719$, Bratu equation has a unique solution. Therefore, equation (18) has a unique solution and for finding that

$$U(x) = -2\ln\left(\frac{\cosh\left([x - 0.5]\frac{\vartheta}{2}\right)}{\cosh\left(\frac{\vartheta}{4}\right)}\right),$$

and ϑ can be derived from

$$\vartheta = \sqrt{2 \ \mu} \ \cosh\left(\frac{\vartheta}{4}\right).$$

As $\mu = 3.513830719,$ so

$$\vartheta = \sqrt{2 \times 3.513830719} \ \cosh\left(\frac{\vartheta}{4}\right). \quad (19)$$

After solving equation (19), we have

$$\vartheta = 4.798645359.$$

Thus,

$$U(x) = -2\ln\left(\frac{\cosh\left([x-0.5]\frac{4.798645359}{2}\right)}{\cosh\left(\frac{4.798645359}{4}\right)}\right),$$

$$\Rightarrow U(x) = -2\ \ln(0.5524420911\cosh(2.399322680\ x - 1.199661340)),$$

which is the exact solution of equation (18). Taking $U_0(x) = 0$ and applying Picard method, we get

$$U_1(x) = 3.999916982\ x - 0.5\ x^2\ ,$$

$$U_2(x) = 0.00130722869\ x^8 - 0.04183044990\ x^7 + 0.4538503540\ x^6 - 1.5225505\ x^5 - 2.196046963\ x^4 - 2.342505194\ x^3 - 1.756915360\ x^2 + 3.999916982\ x\ ,$$

$$U_3(x) = -2.405744959\times 10^{-3}x^{18} + 6.295431774\times 10^{-3}x^{17} + 6.06359428\times 10^{-3}x^{16} - 0.04140070848\ x^{15} + 0.2133926181\ x^{14} - 0.3033346185\ x^{13} - 0.1635341992\ x^{12} - 0.2943933311\ x^{11} - 1.238108848\ x^{10} - 0.09860541020\ x^9 + 0.06549570825\ x^8 + 1.569248472\ x^7 + 2.820111160x^6 - 0.2276930256\ x^5 - 1.827997980\ x^4 - 2.342505194\ x^3 - 1.756915360\ x^2 + 3.999916982\ x, \quad (20)$$

and

$$U_4(x) = 3.999916982\ x - 1.756915360\ x^2 - 2.342505194\ x^3 - 2.608816838\ x^4 + 2.057791678\ x^5 + 1.436367948\ x^6 + 0.3755924260\ x^7 - 0.5989937866\ x^8 - 1.115268983\ x^9 - 0.1861262692\ x^{10} + 0.4014073533x^{11} + 0.4895580669x^{12} - 0.2133778426x^{13} - 0.1783860287x^{14} - 0.2345269362x^{15} - 0.110694331\ x^{16} + 0.0325434749\ x^{17} + 0.0946829798\ x^{18} + 0.0545570354\ x^{19} + 0.023638545x^{20} - 0.0242938671\ x^{21} - 0.021804040\ x^{22} + 0.0072424414\ x^{23} + 0.0741606236\ x^{24} + 0.002463820\ x^{25} - 0.0059534466\ x^{26} - 0.0012045626\ x^{27} - 0.0067346123\ x^{28} - 0.0011077633\ x^{29} + 0.00771358043x^{30} + 0.000802098x^{31} + 0.0003443211x^{32} + 0.61416749\times 10^{-5}\ x^{33} - 0.0201121798\times 10^{-5}\ x^{34} - 0.0922424728\times 10^{-5}\ x^{35} - 0.16190965120\times 10^{-6}\ x^{36} + 0.056308070\times 10^{-6}\ x^{37} + 0.00429760\times 10^{-7}\ x^{38} + 0.0748094200\times 10^{-7}\ x^{39} - 0.0306817000\times 10^{-7}x^{40} - 0.03965400\times 10^{-8}\ x^{41} + 0.05616\times 10^{-9}\ x^{42} - 0.01964754709\times 10^{-9}\ x^{43} + 0.041984642179\times 10^{-9}\ x^{44} - 0.064080398\times 10^{-10}\ x^{45} + 0.07435209\times 10^{-11}\ x^{46} - 0.067626903\times 10^{-12}\ x^{47} + 0.0489542041\times$$

$10^{-13}\ \ x^{48} - 0.0283546 \times 10^{-14}\ \ x^{49} + 0.013108496 \times 10^{-15}\ \ x^{50} - 0.0047865113 \times 10^{-16}\ \ x^{51} + 0.013534696 \times 10^{-17}\ \ x^{52} - 0.0286394 \times 10^{-19}\ \ x^{53} + 0.04273016 \times 10^{-23}\ \ x^{54} - 0.40268047 \times 10^{-25}\ \ x^{55} + 0.17958767 \times 10^{-27}\ \ x^{56}.$

(21)

that is the approximate solution of equation (18).

4.4. Example-4

Consider

$$U''(x) + 5\ \ e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0\ , \quad U(1) = 0\ ; \quad \forall \quad x \in [\ 0\ ,1]. \tag{22}$$

Here, $\mu = 5$. To find the solutions of above equation (22)

$$U(x) = -2\ln\left(\frac{\cosh\left([x - 0.5]\frac{\vartheta}{2}\right)}{\cosh\left(\frac{\vartheta}{4}\right)}\right),$$

and ϑ can be derived from

$$\vartheta = \sqrt{2 \times 5}\ \ \cosh\left(\frac{\vartheta}{4}\right).$$

Since $\mu = 5$, then

$$\vartheta = \sqrt{2 \times 5}\ \ \cosh\left(\frac{\vartheta}{4}\right). \tag{23}$$

After solving equation (23), we get

$$\vartheta = (-3.181805553 + 5.282305940\quad i\ \ )\sqrt{10}.$$

Since ϑ is a complex number, therefore it can not be continued further. It has already been deliberated upon that for $\mu > 3.513830719$, Bratu equation has no solution.

5. CONVERGENCE ANALYSIS

This section discusses the convergence of solutions of numerical examples derived by Picard method.

5.1. Convergence Criteria

The convergence of the sequence $\boldsymbol{U_n(x)}$ generated by Picard's iteration is assessed using the uniform convergence criterion on the interval $\boldsymbol{x} \in [\mathbf{0}, \mathbf{1}]$. Recall that a sequence of functions $\boldsymbol{U_n}(\boldsymbol{x})$ converges uniformly to a function $\boldsymbol{U}(\boldsymbol{x})$ if [35]

$$\lim_{n\to\infty} sup|U_n(x) - U(x)| = 0.$$

It will be demonstrated numerically that for given examples, the maximum absolute error $|U_n(x) - U(x)|$ indeed decreases below a predefined tolerance ε after a finite number of iterations n, providing strong empirical evidence for uniform convergence.

5.2. Convergence Analysis of Solution of Example-1

Since it has been discussed above that $\boldsymbol{U_n}(\boldsymbol{x})$ is uniformly convergent if and only if

$$\sup|\ U_n(x) - U(x)| = \ \| U_n(x) - U(x) \| \ < \varepsilon$$

when $m < n$, $\forall\ x \in [0, 1]$.

Taking $\varepsilon = 0.00009$, we have

$$\sup|\ U_n(x) - U(x)| = \ \| U_n(x) - U(x) \| \ = \text{Max Error} \ < \varepsilon$$

when $m < n$, $\forall\ x \in [0, 1]$

which implies that $U_n(x)$ converges to $U_n(x)$, where $m = 2$ and $U(x)$ represents exact solution.

5.3. Convergence Analysis of Solution of Example-2

Since it has been discussed above that $\boldsymbol{U_n}(\boldsymbol{x})$ is uniformly convergent if and only if

$$\sup|\ U_n(x) - U(x)| = \ \| U_n(x) - U(x) \| \ < \varepsilon$$

when $m < n$, $\forall\ x \in [0, 1]$.

Taking $\varepsilon = 0.0009$, we have

$$\sup|\ U_n(x) - U(x)| = \ \| U_n(x) - U(x) \| \ = \text{Max Error} \ < \varepsilon$$

when $m < n$, $\forall\ x \in [0, 1]$

which implies that $U_n(x)$ converges to $U(x)$, where $m = 2$ and $U(x)$ represents exact solution.

5.4. Convergence Analysis of Solution of Example-3

Since it has been discussed above that $\boldsymbol{U_n}(\boldsymbol{x})$ is uniformly convergent if and only if

$\sup| \; U_n(x) - U(x)| = \; \| U_n(x) - U(x) \| \; < \; \varepsilon$

when $\quad m < \; n \;, \quad \forall \quad x \; \in [\;0\;, 1] \;.$

Taking $\quad \varepsilon \; = \; 0.022414,$ we have

$\sup| \; U_n(x) - U(x)| = \; \| U_n(x) - U(x) \| \; = \text{Max} \quad \text{Error} \; < \varepsilon$

when $\quad m < \; n \;, \quad \forall \quad x \; \in [\;0\;, 1]$

which implies that $U_n(x)$ converges to $U(x)$, where $m = 3$ and $U(x)$ is the exact solution.

6. STABILITY ANALYSIS

This section discusses the stability analysis of numerical scheme for BTEs using the following stability criteria.

6.1. Ulam-Hyers-Rassias Stability (Generalized Ulam-Hyres Stability)

The general second-order differential equation is considered as

$\xi''(x) \; - \; G(x \;, \xi(x) \;, \xi\;'(x)) = 0$ with $\quad \xi(0) = 0$ and $\quad \xi\;'(0) = 0.$ (24) (24)

Following [29-31], it can be said that equation (24) is Ulam-Hyers-Rassias (UHR) stable with respect to a continuous functions $\xi(x)$ if

1. $| \; \xi''(x) \; - \; G(x \;, \xi(x) \;, \; \xi\;'(x))| \leq H(x) \;,$
2. $| \; \xi(x) \; - \; \zeta(x)| \leq M(x) \;,$

where $\xi(x) \;, \zeta(x) \in C^{\,2}[a, b]$ and $H(x), \, M(x)$ are positive continuous functions. Furthermore, $H(x)$ does not depend on $\xi(x)$ and $G(x \;, \xi(x) \;, \xi\;'(x))$.

6.2. Ulam-Hyers-Rassias Stability of Bratu Equation

Theorem 2. The Bratu equation is Ulam-Hyers-Rassias stable, if it fulfills the conditions given below

1. $\left| \; U_n{}''(x) \; + \; \mu e^{U_n(x)} \right| \leq H(x) \;,$
2. $| \; U_n(x) \; - \; U(x)| \leq M(x).$

While a full theoretical proof is beyond the scope of this numerical study, strong numerical evidence will be provided for the stability of derived approximate solutions. For Picard iterations $U_n(x)$, it will be numerically verified that these inequalities hold for given examples.

6.3. Stability Analysis of Example-1

Consider

$$U''(x) + e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0 \ , \quad U(1) = 0 \ ; \quad \forall \ x \in [\,0\,,1].$$

Here, $\xi(x) = U_3(x)$ and $\zeta(x) = U(x)$, then

$$\left| \ U_3''(x) + e^{U_3(x)} \right| \ \leq \ \max\left| \ U_3''(x) + e^{U_3(x)} \right| \ \leq \ H(x) = x^2 + 0.0038857183 \ ,$$

and

$$| \ U_3(x) - U(x) \ | \ \leq \ \max| \ U_3(x) - U(x) \ | \leq \ M(x) = x^2 + 0.000089546 \ .$$

Hence, Example-1 fulfills the hypothesis of Theorem 2 and is Ulam-Hyers-Rassias stable.

6.4. Stability Analysis of Example-2

Consider

$$U''(x) + 2 \ e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0 \ , \quad U(1) = 0 \ ; \quad \forall \ x \in [\,0\,,1].$$

Here, $\xi(x) = U_3(x)$ and $\zeta(x) = U(x)$, then

$$\left| \ U_3''(x) + 2 \ e^{U_3(x)} \right| \ \leq \ \max\left| \ U_3''(x) + 2 \ e^{U_3(x)} \right| \ \leq \ H(x) = x^4 + 0.09701083186 \ ,$$

and

$$| \ U_3(x) - U(x) \ | \ \leq \ \max| \ U_3(x) - U(x) \ | \ \leq \ M(x) = x^4 + 0.00082169 \ .$$

Thus, Example-2 follows the hypothesis of Theorem 2 and is Ulam-Hyers-Rassias stable.

6.5. Stability Analysis of Example-3

Consider

$$U''(x) + 3.513830719 \ e^{U(x)} = 0 \quad \text{with} \quad U(0) = 0 \ , \quad U(1) = 0 \ ; \ \forall \ x \ \in [\,0\,,1].$$

Here, $\xi(x) = U_4(x)$ and $\zeta(x) = U(x)$, then

$$\left| U_4''(x) + 3.513830719 \; e^{U_4(x)} \right| \le \max\left| U_4''(x) + 3.513830719 \; e^{U_4(x)} \right|$$
$$\le H(x) = x^6 + 65.60872438 \; ,$$

and

$$|U_4(x) - U(x)| \le \max|U_4(x) - U(x)| \le M(x)$$
$$= x^6 + 0.022414 \; .$$

Thus, Example-3 fulfils the hypothesis of Theorem 2 and is Ulam-Hyers-Rassias stable.

7. RESULTS AND DISCUSSION

Here, the efficacy and veracity of Picard's iterative scheme has been demonstrated through error's estimation, comparison of maximum errors, and graphical illustrations of the obtained approximate solutions of BTEs.

7.1. Results of Example-1

Table 1 contains $|\mathbf{Error}|$ estimated using Picard method, which shows that $\max |\mathbf{Error}| = \| \mathbf{Error} \| = \mathbf{0.000895460}.$ The $\| \mathbf{Error} \|$ estimated by Picard method, PIA [21], DM [18], HPM [13], and OHAM [15] are listed in Table 2. It explained that Picard's method has smallest $\| \mathbf{Error} \|$.

Figure 2(a , b) demonstrates the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 2(a) shows that both approximate and exact solutions coincide with each other. Figure 2(b) indicates that approximate solutions of zero to third orders are converging to exact solution. Figure 2(c) illustrates the comparison of solutions obtained by Picard's iterative scheme, HPM, OHAM, DM, and PIA with the exact solution. This demonstrates that both exact and Picard's solutions make an excellent agreement, while other solutions diverge from exact solution. Figure 2(d) indicates that Picard's iterative scheme has least $\| \mathrm{Error} \|$ as compared to other stated techniques.

Table 1. Error Estimation of Example-1

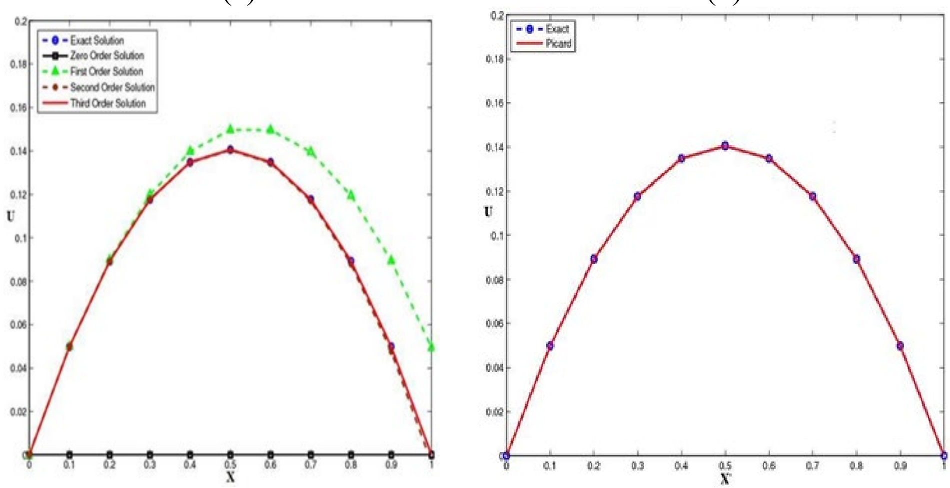
x	Exact Sol.	Picard's Sol.	| Error |
0.00	0.0000000000	0.0000000000	0.0000000000

x	Exact Sol.	Picard's Sol.	| Error |
0.10	0.0498467900	0.0498468027	1.27000×10^{-8}
0.20	0.0891899350	0.0891902388	3.03800×10^{-7}
0.30	0.1176090956	0.1176109762	1.88060×10^{-6}
0.40	0.1347902526	0.1347966401	6.38750×10^{-6}
0.50	0.1405392142	0.1405548425	1.56280×10^{-5}
0.60	0.1347902526	0.1348215219	3.12690×10^{-5}
0.70	0.1176090956	0.1176641754	5.50798×10^{-5}
0.80	0.0891899350	0.0892797819	8.95460×10^{-5}
0.90	0.0498467900	0.0498874514	4.06614×10^{-5}
1.00	0.0000000000	0.0000000000	0.0000000000

Table 2. Comparison of || Error|| of Example-1

Methods	|| Error||
Picard's Method	0.0000895460
Perturbation Iteration Algorithm	0.0011992210
Decomposition Method	0.0030154732
Homotopy Perturbation Method	0.1431299094
OHAM	0.9944206698

(a) (b)



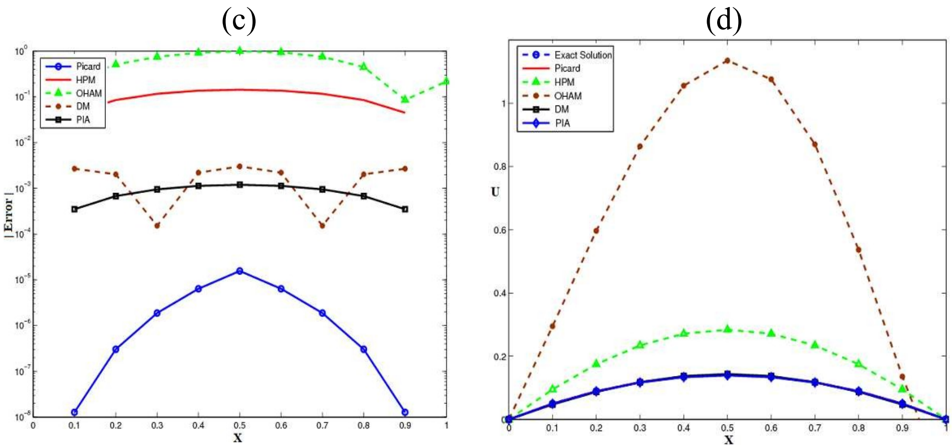


Figure 2. Graphical Illustration of Example-1

7.2. Results of Example-2

Table 3 lists $|\ \mathbf{Error}\ |$ estimated by Picard's iterative scheme. It can be observed that $\|\ \mathbf{Error}\ \| = \mathbf{max}\ |\ \mathbf{Error}\ | = \mathbf{0.00082169}$. The $\|\ \mathbf{Error}\ \|$ estimated by Picard method, Laplace method (LM) [16], DM [18], PIA [14], and HPM [13] drafted in Table 4 indicates that Picard's method has smallest $\|\ \mathbf{Error}\ \|$ amongst these methods.

Figure 3(a , b) illustrated the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 3(a) shows that both approximate and exact solutions match with each other. Figure 3(b) shows that approximate solutions of zero to third orders are converging to exact solution. Figure 3(c) demonstrates the comparison of solutions obtained by Picard's iterative scheme, DM, LM, PIA, and HPM with the exact solution. This illustrates that both exact and Picard's solutions make an excellent agreement, while solutions derived by other mentioned techniques diverge from exact solution. Figure 3(d) represents that Picard's iterative scheme has least ‖ Error ‖ as compared to other stated methods.

Table 3. Error Estimation of Example-2

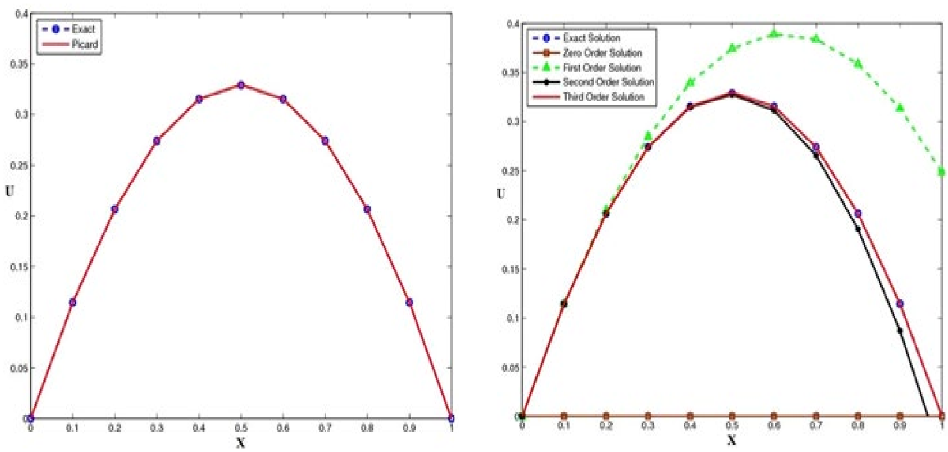
x	Exact Sol.	Picard's Sol.	| Error |
0.00	0.0000000000	0.0000000000	0.000000000
0.10	0.1144107440	0.1144110226	7.7860×10^{-7}

0.20	0.2064191156	0.2064266804	7.5640×10^{-6}
0.30	0.2738793116	0.2739270572	4.7745×10^{-5}
0.40	0.3150893646	0.3152541108	1.6474×10^{-4}
0.50	0.3289524214	0.3293601056	8.2169×10^{-4}
0.60	0.3150893646	0.3152541108	1.6474×10^{-4}
0.70	0.2738793116	0.2739270572	4.7745×10^{-5}
0.80	0.2064191156	0.2064266804	7.5640×10^{-6}
0.90	0.1144107440	0.1144110226	7.7860×10^{-7}
1.00	0.0000000000	0.0000000000	0.000000000

Table 4. Comparison of $\|\text{Error}\|$ of Example-2

Method	$\|\text{Error}\|$
Picard's Method	0.0008216900
Laplace Method	0.0123778084
Decomposition Method	0.0146751156
Perturbation Iteration Algorithm	0.0523780000
Homotopy Perturbation Method	0.2559341111

(a) (b)



(c) (d)

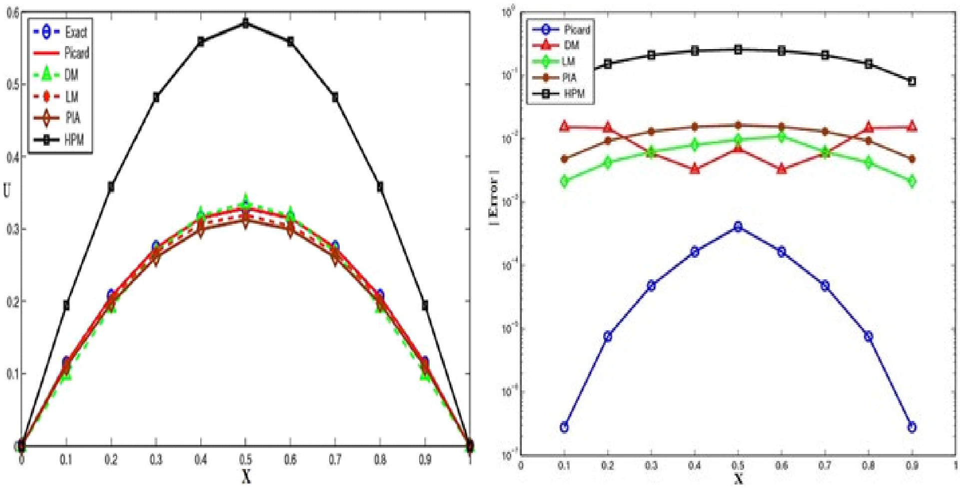


Figure 3. Graphical Illustration of Example-2

7.3. Results of Example-3

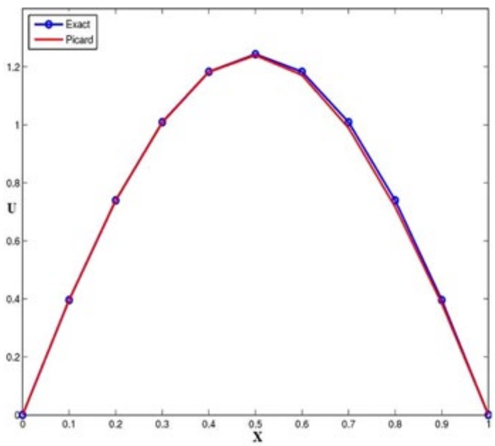
The | **Error** | estimated by Picard's iterative scheme is drafted in Table 5 which shows that $\|\mathbf{Error}\| = \mathbf{max} \; |\mathbf{Error}| = \mathbf{0.022414}$. The $\|\mathbf{Error}\|$ estimated by Picard method, B Spline method [19], HPM [13], PIA [14], and VIM [17] are drafted in Table 6 which shows that Picard method has smallest $\|\mathbf{Error}\|$ amongst these methods.

Figure 4 (a, b) represents the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 4(a) shows that both approximate and exact solutions match with each other. Figure 4(b) shows that approximate solutions of zero to fourth orders are converging to exact solution. Figure 4(c) illustrates a comparison of approximate solutions computed by Picard's iterative scheme, BSM, HPM, PIA, and VIM with exact solution. This demonstrates that both exact and Picard's solutions make an excellent agreement, while solutions computed by other mentioned methods move away from exact solution. Figure 4(d) displays that Picard's iterative scheme has smallest $\|$ Error $\|$ as compared to other stated schemes.

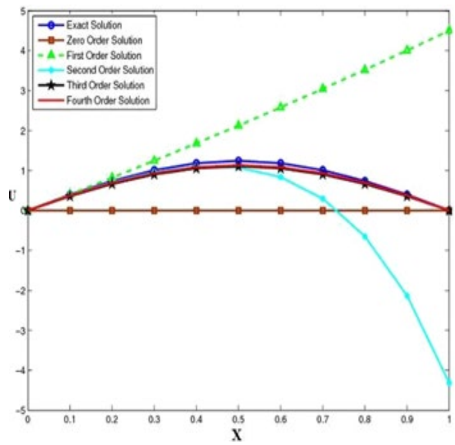
Table 5. Error Estimation of Example-3

x	Exact Sol.	Picard's Sol.	| Error |
0.0	0.000000000	0.0000000000	0.000000000
0.1	0.395805699	0.3957501818	5.5517×10^{-5}
0.2	0.739097410	0.7385461713	5.5124×10^{-4}
0.3	1.008758260	1.0074268710	1.3314×10^{-3}
0.4	1.182536660	1.1817929190	7.4374×10^{-4}
0.5	1.242742690	1.2393843260	3.3584×10^{-3}
0.6	1.182536660	1.1712949450	1.1242×10^{-2}
0.7	1.008758260	0.9890955265	1.9663×10^{-2}
0.8	0.739097410	0.7166833621	2.2414×10^{-2}
0.9	0.395805699	0.3834243783	1.2381×10^{-2}
1.0	0.000000000	0.0000000000	0.000000000

(a)



(b)



(c) (d)

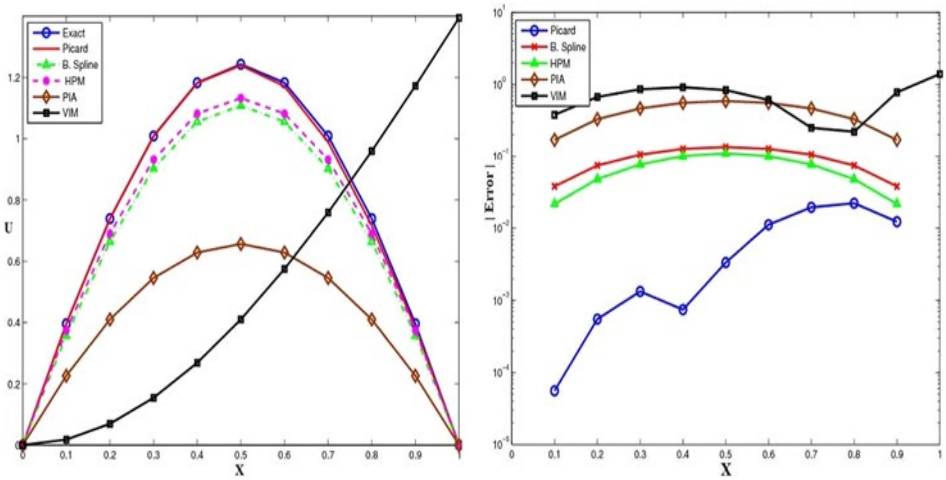


Figure 4. Graphical Illustration of Example-3

Table 6. Comparison of || Error|| of Example-3

Method	||Error||
Picard's Method	0.0224140000
B Spline Method	0.1347528700
Homotopy Perturbation Method	0.4964351282
Perturbation Iteration Algorithm	0.7001783219
Variational Iteration Method	1.3943488610

8. Conclusion

The current article is devoted to the derivation of approximate solutions of BTEs using Picard's iterative technique which is a reliable, effective, and efficient method to solve such kind of equations. Existence and uniqueness of solutions have been confirmed. Convergence and stability of solutions of numerical examples have also been examined. Furthermore, the absolute errors for all examples have also been estimated. The Picard's results are compared with the results attained by DM, PIA, HPM, OHAM, LM, VIM, PM, and BSM through tables and graphical illustrations, and then discussed thoroughly. The derived results definitely designate the following outcomes: a) The solutions of BTEs are convergent, b) The solutions of BTEs are Ulam-Hyers-Rassias stable, c) Picard's solutions have least ‖Error‖ amongst other mentioned techniques, and d) Picard's solutions are in good agreement with exact solutions.

The results of the current research open up a number of promising directions for extension. The Picard's iterative method can be extended to solve higher-dimensional Bratu problems, systems of Bratu equations, and problems with nonlinear boundary conditions. In addition, investigating its application to fractional-order Bratu type equations involving Caputo, Riemann-Liouville, or other fractional derivatives is an important and difficult direction for future study. The stability framework developed here can also be used for other classes of nonlinear differential equations that are solved using iterative methods.

Author Contribution

Saif Ullah: conceptualization, formal analysis, investigation, methodology, project administration, resources, supervision, validation, writing – original draft, writing – reviewing & editing.

Muzaher Ali: data curation, formal analysis, investigation, methodology, software, visualization, writing – original draft, writing – reviewing & editing.

Sana Bajwa: data curation, formal analysis, investigation, methodology, software, visualization, writing – original draft, writing – reviewing & editing.

Ahsan Bilal: data curation, formal analysis, investigation, methodology, writing – original draft, writing – reviewing & editing.

Conflict of Interest

The authors of the manuscript have no financial or non-financial conflict of interest in the subject matter or materials discussed in this manuscript.

Data Availability Statement

The data associated with the study may be provided by the corresponding author if requested.

Funding Details

No funding was received for this research.

Generative AI Disclosure Statement

The authors did not use any type of generative artificial intelligence software for this research.

REFERENCES

1. Wang Y, Yu H, Tan F, Li S. Using an effective numerical method for solving a class of Lane-Emden equations. *Abst Appl Anal*. 2014; 2014: e735831. https://doi.org/10.1155/2014/735831

2. Chu Y. M, Ullah S, Ali M, Tuzzahrah G. F, Munir T. Numerical investigation of Volterra integral equations of second kind using optimal homotopy asymptotic method. *Appl Math Comput*. 2022; 430: e127304. https://doi.org/10.1016/j.amc.2022.127304
3. Saif Ullah, Amin F, Ali M. Numerical investigation with convergence and stability analyses of integro-differential equations of second kind. *Int J Comput Meth*. 2024; 21(3): e2350036. https://doi.org/10.1142/S0219876223500362
4. Mohsin A. A simple solution of Bratu problem. *Comput Math Appl*. 2014; 67: 26-33. https://doi.org/10.1016/j.camwa.2013.11.004
5. Gelfand I. M. Some problems in the theory of quasi-linear equations. *Transac Amer Math Soc*. 1963; 29: 295-381.

   https://doi.org/10.1090/S0002-9947-1963-0153125-6
6. Wan YQ, Guo Q, Pan N. Thermo electro hydrodynamic model for electrospinning process. *Int J Nonlin Sci Numer Simul*. 2004; 5(1): 5-8. https://doi.org/10.1515/IJNSNS.2004.5.1.5
7. Jacobsen J, Schmitt K. The Liouville-Bratu-Gelfand problem for radial operators. *J Differ Equ*. 2002; 184: 283-298. https://doi.org/10.1006/jdeq.2001.4151
8. Kafri HQ, Khuri SA. A novel approach using fixed point iterations and Green's functions. *Comput Phy Commun*. 2016; 198: 97-104. https://doi.org/10.1016/j.cpc.2015.09.007
9. Abukhaled M, Khuri SA, Sayfy A. Spline-based numerical treatments of Bratu-type equations. *Palestine J Math*. 2012; 1: 63-70.
10. Boyd JP. Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one-dimensional Bratu equation. *Appl Math Comput*. 2003; 142: 189-200. https://doi.org/10.1016/S0096-3003(02)00293-8
11. WazwazA M. Adomian decomposition method for a reliable treatment of the Bratu-type equations. *Appl Math Comput*. 2005; 166: 652-663. https://doi.org/10.1016/j.amc.2004.06.059
12. Kashkari BSH, Abbas SZ. Solution of initial value problem of Bratu type equation using modifications of homotopy perturbation method.

Int J Comput Appl. 2017; 162: 44-49. https://doi.org/10.5120/ijca2017913377

13. Feng X, He Y, Meng Y. Application of homotopy perturbation method to the Bratu type problems. *Topol Meth Nonlin Anal: J Juliusz Schauder Cent*. 2008; 31: 243-252. https://doi.org/10.12775/TMNA.2008.015
14. Aksoy Y, Pakdemirli M. New perturbation iteration solutions for Bratu-type equations. *Comput Math Appl*. 2010; 59: 2802-2808. https://doi.org/10.1016/j.camwa.2010.01.050
15. Darwish MA, Kashkari BS. Numerical solutions of second order initial value problems of Bratu type via optimal homotopy asymptotic method. *Am J Comput Math*. 2014; 4: 47-54. https://doi.org/10.4236/ajcm.2014.41004
16. Khuri SA. A new approach to Bratu's problem. *Appl Math Comput*. 2004; 147: 131-136. https://doi.org/10.1016/S0096-3003(02)00655-2
17. Batiha B. Numerical solutions of Bratu type equations by the variational iteration method. *Hacett J Math Statis*. 2010; 391: 23-29.
18. Salem SA, Thanoon TY. On solving Bratu's type equation by perturbation method. *Int J Nonlin Anal Appl*. 2022; 13(1): 2755-2763. https://doi.org/10.22075/IJNAA.2022.5983
19. Caglar H, Caglar N, Ozer M, Valaristos A, Anagnostopoulos AN. B Spline method for solving Bratu's problem. *Int J Comput Math*. 2010; 87(8): 1885-1891. https://doi.org/10.1080/00207160802450136
20. Gutierrez JM, Veron MAH. A picard-type iterative scheme for Fredholm integral equations of the second kind. *Mathematics*. 2021; 9(1): 1-15. https://doi.org/10.3390/math9010087
21. Robin W. Iterative solutions to classical second-order ordinary differential equations. *J Innov Technol Edu*. 2019; 6: 1-12. https://doi.org/10.69437/jite.6.1_1
22. Rahman M. *Integral Equations and their Applications*. Southhampton: WIT Press; 2007.
23. Wang Y, Ni G, Liu Y. Multi step Newton-Picard method for nonlinear differential equations. *J Guid Cont Dynam*. 2020; 43(11): 2148-2155. https://doi.org/10.2514/1.G005249

24. Yang AM, Zhang C, Jafari H, Cattani C, Jiao Y. Picard successive approximation method for solving differential equations arising in fractal heat transfer with local fractional derivative. *Abstract Appl Anal*. 2014; 2014: e395710. https://doi.org/10.1155/2014/395710

25. Ulam S. M. *A Collection Mathematical Problems*. Interscience; 1960.

26. Hyers D, H. On the stability of the linear functional equation. *Proc Natl Acad Sci*. 1941; 27(4): 222-224. https://doi.org/10.1073/pnas.27.4.222

27. Zhao K. Generalized UH-stability of a nonlinear fractional coupling (p1, p2)-Laplacian system concerned with nonsingular Atangana-Baleanu fractional calculus. *J Inequal Appl*. 2023; 2023: e100. https://doi.org/10.1186/s13660-023-03009-w

28. Zhao K. Solvability and GUH-stability of a nonlinear CF-fractional coupled Laplacian equations. *AIMS Mathematics*. 2023; 8(6): 13351-13367. https://doi.org/10.3934/math.2023673

29. Zada A, Faisal S, Li Y. Hyers-Ulam-Rassias stability of nonlinear delay differential equations. *J Nonl Sci Appl*. 2017; 10: 504-510.

https://doi.org/10.22436/jnsa.010.02.08

30. Huang J, Jung S. M, Li Y. On Hyers-Ulam stability of nonlinear differential equations. *Bull Korean Math Soc*. 2015; 52: 685-697.

https://doi.org/10.4134/BKMS.2015.52.2.685

31. Qarawani MN. Hyers-Ulam-Rassias stability criteria of nonlinear differential equations of Lane-Emden type. *Gen Lett Math*. 2017; 3(1): 57-70. https://doi.org/10.31559/glm2017.3.1.6

32. Wang X, Luo D, Zhu Q. Ulam-Hyers stability of Caputo type fuzzy fractional differential equations with time-delays. *Chaos Solit Fract*. 2022; 156: e111822. https://doi.org/10.1016/j.chaos.2022.111822

33. Luo D, Wang X, Caraballo T, Zhu Q, Ulam-Hyers stability of Caputo-type fractional fuzzy stochastic differential equations with delay. *Commun. Nonlin Sci Numer Simul*. 2023; 121: e107229. https://doi.org/10.1016/j.cnsns.2023.107229

34. Wazwaz AM. *A First Course in Integarl Equations*. World Scientific Publishing Company; 2015.

35. Reddy BD. *Introductory Functional Analysis With Application to Boundary Value Problems And Finite Elements*. Springer Verlag Incorporation; 1998.