CONVERGENCE AND ULAM-HYERS-RASSIAS STABILITY ANALYSES WITH ERROR ESTIMATION OF NUMERICAL SOLUTIONS OF BRATU TYPE EQUATIONS

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ABSTRACT. A detailed investigation of properties and numerical solutions of Bratu type equations is performed in this article. The approximate solutions of Bratu differential equations are computed by Picard's iterative scheme. The existence, uniqueness and convergence of solutions are verified for every example. It has also been proved that Bratu type equations are Ulam-Hyers-Rassias stable. Furthermore, the derived results are compared with those already exist in the literature so as to validate that Picard's scheme estimates the actual solutions very precisely as compared to other numerical techniques. The absolute errors are also estimated and compared which verified that results obtained by Picard's method have least error.

Keywords: Bratu type equations, Picard's method, existence and uniqueness of solutions, convergence analysis, Ulam-Hyers-Rassias stability, error estimation.

Abbreviations

BTEs	Bratu type equations	ODE	Ordinary differential equation
DM	Decomposition method	VIM	Variational Iteration method
PM	Perturbation method	HPM	Homotopy Perturbation method
LM	Laplace method	PIA	Perturbation Iteration Algorithm
BSM	B Spline method	OHAM	Optimal Homotopy Asymptotic method

1. Introduction

Mathematical modeling of many physical phenomena achieved in the form of integral equations, differential equations, or integro-differential equations [1–3]. Bratu type differential equations have many applications in practical life including fuel ignition, Chandrasekhar model, nanotechnology, thermal reaction, heat transfer by conduction, radiative heat transfer, and chemical reactor theory [4–7]. The scientists and researchers are very keen in the study of Bratu type equations (BTEs) as the solutions of such types of equations are more natural and significant because of their important usage in engineering and scientific experimentations.

Following equation is called one dimensional Bratu equation [8, 9]

$$U^{''}(x) + \mu e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$ (1)

where $\mu > 0$ is a physical parameter. Exact solution of above equation (1) is [10]

$$U(x) = -2 \ln \frac{\cosh \left[x - 0.5\right] \frac{\vartheta}{2}}{\cosh\left(\frac{\vartheta}{4}\right)}$$

where θ can be obtained from

$$\theta = \sqrt[3]{\frac{1}{\mu}} \cosh \frac{\theta}{4}$$

Several analytical, semi-analytical, and numerical schemes have been used to solve BTEs. Decomposition method (DM) [11], Homotopy Perturbation method (HPM) [12, 13], Perturbation Iteration Algorithm (PIA) [14], Optimal Homotopy Asymptotic method (OHAM) [15], Laplace method (LM) [16], Variational Iteration method (VIM) [17], Perturbation method (PM) [18], B Spline method (BSM) [19] have been implemented to handle Bratu type equations (BTEs).

Picard's method is an efficient and accurate technique which is used to solve nonlinear differential equations, linear and nonlinear integral equations, and differential equations arising in fractal heat transfer [20–24]. It converges rapidly and gives accurate solution, especially when it is applied to nonlinear equation. This aspect motivates us to solve Bratu equations using Picard's method.

The study of the Ulam-Hyers type stability of various dynamic systems has received great attention. The famous Ulam-Hyers stability originated in a lecture at the Wisconsin University in 1940, in which Ulam [25] proposed the problem of the stability for functional equations. A significance breakthrough came in 1941, when Hyers [26] gave a partial solution to Ulam's problem. After in-depth analysis of the Ulam-Hyers stability structure, some researchers have extended and used the concept of Ulam-Hyers stability, such as generalized Ulam-Hyers stability [27, 28], Ulam-Hyers-Rassias stability [29], generalized Ulam-Hyers-Rassias stability, etc. Huang et al. [30] analyzed Hyers-Ulam stability while Qarawani [31] investigated Hyers-Ulam-Rassias stability of nonlinear differential equations. Wang et al. considered Ulam-Hyers stability of fuzzy fractional differential equations with delay [32, 33].

The foremost aim of present article is to compute approximate solutions of BTEs using Picard's method which are new and not described before in the literature. The article is organized as: Section 2 covers conditions for existence and uniqueness of solutions, while section 3 comprehend the illustration of Picard's method. Section 4 has solutions of numerical examples. The convergence and stability analyses of derived solutions have been performed in section 5 and 6, respectively. Error estimation and comparison of results are incorporated in section 7, while the article is concluded in section 8.

2. Conditions for Existence and Uniqueness of Solutions of Bratu Equations

Below given result provides fundamental conditions for existence and uniqueness of solutions of Bratu equations.

Theorem 1. There are three cases for μ given in equation (1)

- If $\mu = \mu_0$, then there exists one solution,
- If $\mu < \mu_0$, then there exist two solutions,
- If $\mu > \mu_0$, then no solution exists.

Here μ_0 is called the critical value [12,13] given as

$$1 = \frac{1}{4} \sqrt{2\mu_0} \sinh \frac{\theta}{4} ,$$

and its numerical value is

$$\mu_0 = 3.513830719.$$

Figure 1 shows that when μ < 3.513830719, then θ has two values for each value of μ . Hence, there are two solutions for each μ < 3.513830719. For μ = 3.513830719, only one value of θ = 4.798645359 exists and we have unique solution. When μ > 3.513830719, we do not have resultant value of θ , and in this case, no solution of Bratu equation exists [12,13].

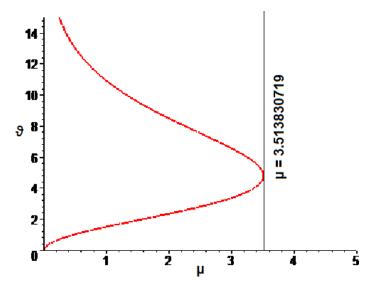


Figure 1: Comparison of μ and θ .

3. DESCRIPTION of PICARD'S METHOD

Description of Picard's method for second order ordinary differential equation (ODE) and its implementation to Bratu equation are carried out in this section.

3.1. Formulation of Picard's Method for Second Order ODE. Let us take a second order ODE

$$L(U(x)) = N(U(x)), \tag{2}$$

where L(U(x)) is linear operator while N(U(x)) is nonlinear operator [20].

Writing L(U(x)) in terms of derivative of second order as

$$L(U(x)) = U(x),$$

while N(U(x)) in terms of U(x) and U(x) as

$$\mathsf{N}(U(x)) = \ U(x), \ U^{'}(x) \ ,$$

then, equation (2) yields

$$U_{..}(x) = N \quad U(x), \quad U_{.}$$
 (3)

where

$$U(0) = \alpha$$
 and $U(0) = \beta$.

Solving equation (3), we have

$$U(x) = \alpha + \beta x + \begin{cases} J^{x} \\ N \\ 0 \end{cases} 0$$
 N $U(x)$, $U'(x)$ $dxdx$. (4)

Hence, the iterative formula is

nula is
$$\int_{x} x$$

$$U(x) = \alpha + \beta x + \int_{x} 0$$

$$N(x) = \int_{x} 0$$

$$U_{n}(x), U_{n}(x) dx dx,$$

$$U_{n}(x) = \int_{x} 0$$

$$U_{n}(x) dx dx,$$

$$U_{n}(x) = \int_{x} 0$$

$$U_{n}(x) dx dx,$$

$$U_{n}(x) dx dx,$$

that is Picard's iterative scheme for second order ordinary differential equation [21, 22].

3.2. Implementation to Bratu Equation. Let us take equation (1) as

$$U'(x) + \mu e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$. (6)

Here

$$L(U(x)) = U'(x)$$
 and $N(U(x)) = \mu e^{U(x)}$.

From equation(6), we have [12, 34]

$$U''(x) = -\mu e^{U(x)}. (7)$$

Implementation of Picard's method, yields

$$U_{n+1}(x) = \alpha + \theta x - \mu e^{Un(x)} dx dx,$$
(8)

where

$$U(0)=\alpha=0,$$

and θ changes as μ changes

$$\Rightarrow U_{n+1}(x) = \beta x - \mu e^{U_n(x)} dx dx.$$
(9)

This is Picard's iterative scheme for Bratu equation.

4. Solutions of BTEs using Picard's Method

In this section, we use Picard's iterative scheme to compute approximate solutions of various Bratu type equations.

4.1. Example 1. Consider

$$U''(x) + e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$. (10)

Here, μ = 1. As previously deliberated that for μ < 3.513830719, there exist two solutions of Bratu equation. Thus, equation (10) has two solutions and for finding those solutions

$$U(x) = -2 \ln \frac{\cosh [x - 0.5] \frac{3}{2}}{\cosh (\frac{3}{4})}$$

where θ can be obtained from

$$\theta = \sqrt[4]{\frac{1}{\mu}} \cosh \frac{\theta}{4}$$
.

Since $\mu = 1$, so

$$\theta = \sqrt{2} \cosh \frac{\theta}{4} . \tag{11}$$

After solving equation (11), we have

$$\theta_1 = 1.517164598$$
, $\theta_2 = 10.9387028$.

Thus, we obtained two solutions θ_1 (called lower solution) and θ_2 (called higher solution). However in the present case, merely lower solution will be used, then

$$U(x) = -2 \ln \frac{\cosh \left[x - 0.5\right] \frac{1.517164598}{2}}{\cosh \frac{1.517164598}{4}},$$

$$\Rightarrow$$
 $U(x) = -2 \ln 0.9321424738 \cosh (0.7585822990 x - 0.3792911495),$

that is the exact solution of equation (10). Taking $U_0(x) = 0$ and applying Picard's method, we get

$$U_1(x) = 0.5493527280 x - 0.5 x^2,$$

$$U_2(x) = 0.5493527280 x - 0.5 x^2 - 0.09155878800 x^3 + 0.02909214918 x^4 + 0.01235224910 x^5$$

$$- 0.001651763169 x^6 - 0.001634978357 x^7 + 0.0003720238095 x^7,$$
(12)

and

$$U_3(x) = 0.5493527280 x - 0.5 x^2 - 0.09155878800 x^3 + 0.02909214918 x^4 + 0.01831175760 x^5$$

$$- 0.003459802640 x^6 - 0.001764607014 x^7 + 0.00009322482480 x^8 + 0.0001580853532 x$$

$$+ 0.4533942348 \times 10^{-5} x^{10} - 0.00001393134891 x^{11} + 0.06120893625 \times 10^{-6} x^{12}$$
(13)

$$+0.06540389326 \times 10^{-5} x^{13} + 0.04400289 \times 10^{-6} x^{14} - 0.03474251 \times 10^{-6} x^{15}$$

$$-0.03008674 \times 10^{-7} x^{16} + 0.02236216459 \times 10^{-7} x^{17} - 0.02261465929 \times 10^{-8} x^{18}$$

that is the approximate solution of equation (10).

4.2. Example 2. Consider

$$U(x)U(x) + 2e = 0$$
 with $U(0) = 0$, $U(1) = 0$, $V(1) = 0$

Here, μ = 2. As previously deliberated that for μ < 3.513830719, there exist two solutions of Bratu equation. Thus, equation (14) has two solutions and for finding those solutions

$$U(x) = -2 \ln \frac{\cosh \left[x - 0.5\right] \frac{9}{2}}{\cosh(\frac{9}{4})},$$

where θ can be derived from

$$\theta = \sqrt[4]{2\mu} \cosh \frac{\theta}{4} .$$

Since $\mu = 2$, so

$$\theta = \sqrt{2 \times 2} \cosh \frac{\theta}{4} . \tag{15}$$

After solving equation (15), we get

$$\theta_1 = 2.357551054$$
, $\theta_2 = 8.5071995$.

Thus, we obtained two solutions θ_1 (called lower solution) and θ_2 (called higher solution). However in the present case, merely lower solution will be used, then

$$U(x) = -2 \ln \frac{\cosh \left[(x - 0.5) \right] \frac{2.357551054}{2}}{\cosh \frac{2.357551054}{4}},$$

 \Rightarrow $U(x) = -2 \ln 0.8483379380 \cosh (1.178775527 x - 0.5893877635) ,$

that is the exact solution of equation (14). Taking $U_0(x) = 0$ and applying Picard's method, we get

$$U_1(x) = 1.248217518 x - x^2,$$

$$U_2(x) = 1.248217518 x - x^2 - 0.4160725060 x^3 + 0.03682941898 x^4 + 0.09240872640 x^5$$
(16)

and

$$U_3(x) = 1.248217518 x - x^2 - 0.4160725060 x^3 + 0.03682941898 x^4 + 0.1664290024 x^5 - 0.001165361882 x^6 - 0.02640249326 x^7 - 0.006559875108 x^8 + 0.003173141654 x^9$$

$$+0.001944801524 x^{10} - 0.0005966016460 x^{11} - 0.0001722400368 x^{12}$$
 (17)

$$+0.00002374648306 x^{13} + 0.00002586929702 x^{14} - 0.02644413 \times 10^{-6} x^{15}$$

 $+ 0.01860156574 x^6 - 0.02971946472 x^7 + 0.005952380952 x^8$

$$-0.4602890 \times 10^{-5} x^{16} + 0.1300746880 \times 10^{-5} x^{17} - 0.1157870556 \times 10^{-6} x^{18}$$

that is the approximate solution of equation (14).

4.3. Example 3. Consider

$$U(x)U(x) + 3.513830719 e = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$. (18)

Here, $\mu = 3.513830719$. As previously deliberated that for $\mu = 3.513830719$, Bratu equation has a unique solution. Therefore, equation (18) has a unique solution and for finding that

$$U(x) = -2 \ln \frac{\cosh \left[x - 0.5\right] \frac{3}{2}}{\cosh(\frac{3}{4})},$$

and heta can be derived from

$$\theta = \sqrt[4]{2\mu} \cosh \frac{\theta}{4} .$$

As $\mu = 3.513830719$, so

$$\theta = \sqrt[4]{2 \times 3.513830719} \cosh \frac{\theta}{4}$$
 (19)

After solving equation (19), we have

$$\theta = 4.798645359.$$

Thus,

$$U(x) = -2 \ln \frac{\cosh \left[x - 0.5\right] \frac{4.798645359}{2}}{\cosh \frac{4.798645359}{4}}$$

$$\Rightarrow$$
 $U(x) = -2 \ln 0.5524420911 \cosh (2.399322680 x - 1.199661340) ,$

that is the exact solution of equation (18). Taking $U_0(x) = 0$ and applying Picard's method, we get $U_1(x) = 3.999916982 x - 0.5 x^2$

$$U_{2}(x) = 3.999916982 x - 1.756915360 x^{2} - 2.342505194 x^{3} - 2.196046963 x^{4}$$

$$- 1.522550589 x^{5} + 0.4538503540 x^{6} - 0.04183044990 x^{7} + 0.00130722869 x^{8},$$

$$U_{3}(x) = 3.999916982 x - 1.756915360 x^{2} - 2.342505194 x^{3} - 1.827997980 x^{4} - 0.2276930256 x^{5}$$

$$+ 2.820111160 x^{6} + 1.569248472 x^{7} + 0.06549570825 x^{8} - 0.09860541020 x^{9} - 1.238108848 x^{10}$$

$$- 0.2943933311 x^{11} - 0.1635341992 x^{12} - 0.3033346185 x^{13} + 0.2133926181 x^{14}$$

 $-0.04140070848 x^{15} + 0.006063594280 x^{16} + 0.006295431774 x^{17} - 0.002405744959 x^{18}$ and

$$U_4(x) = 3.999916982 \, x - 1.756915360 \, x^2 - 2.342505194 \, x^3 - 2.608816838 \, x^4 + 2.057791678 \, x^5 \\ + 1.436367948 \, x^6 + 0.3755924260 \, x^7 - 0.5989937866 \, x^8 - 1.115268983 \, x^9 - 0.1861262692 \, x^{10} \\ + 0.4014073533 \, x^{11} + 0.4895580669 \, x^{12} - 0.2133778426 \, x^{13} - 0.1783860287 \, x^{14} - 0.2345269362 \, x^{15} \\ - 0.1108694331 \, x^{16} + 0.03254347491 \, x + 0.09468297982 \, x + 0.05455703540 \, x \\ + 0.002336385453 \, x^{20} - 0.02429386712 \, x^{21} - 0.02180400406 \, x^{22} + 0.007242441400 \, x^{23} \\ + 0.007416062365 \, x^{24} + 0.0024638204 \, x^{25} - 0.0005953446661 \, x^{26} - 0.001204562610 \, x^{27} \\ - 0.0006734612338 \, x^{28} - 0.0001107763731 \, x^{29} + 0.0007713580433 \, x + 0.000080209802 \, x \\ + 0.0000344321121 \, x^{32} + 0.6141674 \, \times \, 10^{-5} \, x^{33} - 0.02011121798 \, \times \, 10^{-5} \, x^{34} \\ - 5$$

$$-0.0922424728 \times 10 \quad x^{35} - 0.1619096512 \times 10^{-6} \quad x^{36} + 0.05630807 \times 10^{-6} \quad x^{37} \\ +0.0042976 \times 10^{-7} \quad x^{3} + 0.07480942 \times 10^{-7} \quad x^{39} - 0.0306817 \times 10^{-7} \quad x^{40} - 0.039654 \times 10^{-8} \quad x^{41} \\ +0.05616 \times 10^{-9} \quad x^{42} - 0.01964754709 \times 10^{-9} \quad x^{43} + 0.04198464217 \times 10^{-9} \quad x^{44} \\ -0.06408030986 \times 10^{-10} \quad x^{45} + 0.07435209057 \times 10^{-11} \quad x^{49} + 0.01310784960 \times 10^{-15} \quad x^{50} \\ +0.04895422041 \times 10^{-13} \quad x^{48} - 0.0283546 \times 10^{-14} \quad x^{9} + 0.01310784960 \times 10^{-15} \quad x^{50} \\ -16 \quad -17 \quad 52 \quad -17 \quad 52 \quad -19 \quad -$$

$$-0.004786551130\times 10 \quad \underset{54}{x} + 0.001353426966\times 10^{-17} \quad \underset{55}{x} - 0.00286394\times 10 \quad \underset{-27}{x} \quad \underset{56}{x} + 0.04275830164\times 10^{-23} \quad \underset{54}{x} - 0.4022680475\times 10^{-25} \quad \underset{55}{x} + 0.1795876770\times 10^{-27} \quad \underset{56}{x} \quad . \tag{21}$$

that is the approximate solution of equation (18).

4.4. Example 4. Consider

$$U(x) + 5e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$. (22) Here, $\mu = 5$. To find the solutions of above equation (22)

$$U(x) = -2 \ln \frac{\cosh [x - 0.5] \frac{3}{2}}{\cosh(\frac{3}{4})}$$

and θ can be derived from

$$\theta = \sqrt[4]{2 \times 5} \cosh \frac{\theta}{4} .$$

$$\theta = \sqrt[4]{2 \times 5} \cosh \frac{\theta}{4} .$$

Since $\mu = 5$, then

$$\theta = \sqrt[4]{2 \times 5} \cosh \frac{\theta}{4} . \tag{23}$$

After solving equation (23), we get

$$\theta = (-3.181805553 + 5.282305940 i)^{\sqrt{10.}}$$

Since θ is a complex number, therefore we can not continue further. We have deliberated already that for $\mu > 3.513830719$, Bratu equation has no solution.

5. Convergence Analysis

The convergence of solutions of numerical examples derived by Picard's method is discussed in this section.

5.1. **Convergence Criteria.** Let $U_n(x)$ be the sequence of functions approximated by Picard's method defined on set $D \subset \mathbb{R}^n$. The sequence $U_n(x) \to U(x)$ when for each $\varepsilon > 0$, there ex- ist $m, n \in \mathbb{Z}^+$ for which [35]

$$\sup |U_n(x) - U(x)| = ||U_n(x) - U(x)|| < \varepsilon \quad \text{when} \quad m < n.$$

When m and ε depend upon x, the convergence is called pointwise convergence, while the convergence is said to be uniform convergence when m and ε do not depend on x. In the limiting form, it is represented as [35]

$$\lim_{n\to\infty}U_n(x)=U(x).$$

Theorem 2. The sequence $U_n(x)$ is uniformly convergent iff

$$\lim_{n\to\infty}||U_n(x)-U(x)||=0.$$

5.2. Convergence of Solution of Example 1. As we have discussed in Theorem 2 that $U_n(x)$ is convergent uniformly iff

$$\sup |U_n(x) - U(x)| = ||U_n(x) - U(x)|| < \varepsilon \text{ when } m < n, \quad \forall x \in [0, 1].$$

Taking

$$\varepsilon = 0.00009$$
,

we have

sup $U_n(x) - U(x) = ||U_n(x) - U(x)|| = \text{Max Error} < \varepsilon$ when m < n, $\forall x \in [0, 1]$ which implies that $U_n(x)$ converges to U(x), where m = 2 and U(x) represents exact solution.

5.3. Convergence of Solution of Example 2. As we have discussed in Theorem 2 that $U_n(x)$ is convergent uniformly iff

$$\sup |U_n(x) - U(x)| = ||U_n(x) - U(x)|| < \varepsilon \text{ when } m < n, \quad \forall x \in [0, 1].$$

Taking

$$\varepsilon = 0.0009$$
,

we have

sup $U_n(x) - U(x) = ||U_n(x) - U(x)|| = \text{Max Error} < \varepsilon$ when m < n, $\forall x \in [0, 1]$ which implies that $U_n(x)$ converges to U(x), where m = 2 and U(x) represents exact solution.

5.4. Convergence of Solution of Example 3. As we have discussed in Theorem 2 that $U_n(x)$ is convergent uniformly iff

$$\sup U_n(x) - U(x) = ||U_n(x) - U(x)|| < \varepsilon \text{ when } m < n, \quad \forall x \in [0, 1].$$

Taking

$$\varepsilon = 0.022414$$
,

we have

sup
$$U_n(x) - U(x) = ||U_n(x) - U(x)|| = \text{Max Error} < \varepsilon$$
 when $m < n$, $\forall x \in [0, 1]$ which implies that $U_n(x)$ converges to $U(x)$, where $m = 3$ and $U(x)$ is the exact solution.

6. Stability Analysis

We perform stability analysis of numerical scheme for Bratu type equations by using the following stability criteria.

6.1. **Generalized Ulam-Hyres Stability (Ulam-Hyers-Rassias Stability).** Let the second order differential equation is

$$\xi_{0}(x) - G(x), \xi_{0}(x) = 0$$
 with $\xi(0) = 0$ and $\xi_{0}(0) = 0$. (24)

Theorem 3. The above equation (24) is Ulam-Hyers-Rassias stable, if it fulfills the conditions given below

(1)
$$\xi_{-}(x) - G(x), \xi_{-}(x) \leq H(x),$$

$$(2) \quad \xi(x) - \zeta(x) \leq M(x),$$

where $\xi(x)$, $\zeta(x) \in C^2[a,b]$ and H(x), M(x) are positive continuous functions. Furthermore, H(x) does not depend on $\xi(x)$ and G(x), $\xi(x)$, $\xi(x)$

6.2. Stability of Example 1. Consider

$$U'(x) + e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$.

Here, $\xi(x) = U_3(x)$ and $\zeta(x) = U(x)$, then

$$U_3^{''}(x) + e^{U_3(x)} \le \max U_3^{''}(x) + e^{U_3(x)} \le H(x) = x^2 + 0.0038857183,$$

and

$$|U_3(x) - U(x)| \le \max |U_3(x) - U(x)| \le M(x) = x^2 + 0.000089546$$
.

Hence, Example 1 fulfills the conditions of Theorem 3 and is Ulam-Hyers-Rassias stable.

6.3. Stability of Example 2. Consider

$$U(x) + 2e^{U(x)} = 0$$
 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$.

Here, $\xi(x) = U_3(x)$ and $\zeta(x) = U(x)$, then

$$U_3^{''}(x) + 2e^{U_3(x)} \le \max U_3^{''}(x) + 2e^{U_3(x)} \le H(x) = x^4 + 0.09701083186,$$

and

$$|U_3(x) - U(x)| \le \max |U_3(x) - U(x)| \le M(x) = x^4 + 0.00082169.$$

Thus, Example 2 follows the conditions of Theorem 3 and is Ulam-Hyers-Rassias stable.

6.4. Stability of Example 3. Consider

$$U(x)U(x) + 3.513830719 e$$
 = 0 with $U(0) = 0$, $U(1) = 0$; $\forall x \in [0, 1]$.

Here, $\xi(x) = U_4(x)$ and $\zeta(x) = U(x)$, then

$$U_4(x)$$
 $U_4(x)$ 6
 $U_4(x) + 3.513830719 e$ $\leq \max U_4(x) + 3.513830719 e$ $\leq H(x) = x + 65.60872438$,

and

$$|U_4(x) - U(x)| \le \max |U_4(x) - U(x)| \le M(x) = x^6 + 0.022414$$
.

Thus, Example 3 satisfies the conditions of Theorem 3 and is Ulam-Hyers-Rassias stable.

7. RESULTS and DISCUSSIONS

Here, we demonstrate the efficacy and veracity of Picard's iterative scheme through error's estimation, comparison of maximum errors and graphical illustrations of the obtained approximate solutions of BTEs.

7.1. **Results of Example 1.** Table 1 contains Erron estimated by using Picard's method, which shows that max Error = ||Erron|| = 0.000895460. The ||Error|| estimated by Picard's method, Perturbation Iteration Algorithm (PIA) [21], Decomposition method (DM) [18], HPM (Homotopy Perturbation method) [13], and OHAM [15] are listed in Table 2 which indicates that Picard's method has smallest ||Error||.

Figure 2(a, b) demonstrates the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 2(a) shows that both approximate and exact solutions coincide with each other. Figure 2(b) shows that approximate solutions of zero to third orders are converging to exact solution. Figure 2(c) illustrates the comparison of solutions obtained by Picard's iterative scheme, HPM, OHAM, DM, and PIA with the exact solution, which shows that Picard's solution and exact solution make an excellent agreement while other solutions diverge from exact solution. Figure 2(d) shows that Picard's iterative scheme has least $\|\text{Error}\|$ as compared to other mentioned techniques.

Table 1: Error Estimation of Example 1

X	Exact Solution	Picard's Solution	Error
0.00	0.0000000000	0.000000000	0.0000000000
0.10	0.0498467900	0.0498468027	1.27000 × 10
0.20	0.0891899350	0.0891902388	3.03800 × 10 -
0.30	0.1176090956	0.1176109762	1.88060 × 10
0.40	0.1347902526	0.1347966401	6.38750×10^{-6}
0.50	0.1405392142	0.1405548425	1.56280 × 10
0.60	0.1347902526	0.1348215219	3.12690 × 10
0.70	0.1176090956	0.1176641754	5.50798 × 10
0.80	0.0891899350	0.0892797819	8.95460 × 10
0.90	0.0498467900	0.0498874514	4.06614 × 10
1.00	0.0000000000	0.000000000	0.0000000000

Table 2: Comparison of ||Error|| of Example 1

Methods	Error
Picard's method	0.0000895460
Perturbation Iteration Algorithm	0.0011992210
Decomposition method	0.0030154732
Homotopy Perturbation method	0.1431299094
OHAM	0.9944206698

7.2. **Results of Example 2.** Table 3 lists Error estimated by Picard's iterative scheme. We can observe that $||\text{Error}|| = \max \text{Error} \neq 0.00082169$. The ||Error|| estimated by Picard's method, Laplace method (LM) [16], Decomposition method (DM) [18], Perturbation Iteration Algorithm (PIA) [14], and Homotopy Perturbation method (HPM) [13] contained in Table 4 indicates that Picard's method has smallest ||Error|| amongst these methods.

Figure 3(a, b) illustrates the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 3(a) shows that both approximate and exact solutions match with each other. Figure 3(b) shows that approximate solutions of zero to third orders are converging to exact solution. Figure 3(c) demonstrates the comparison of solutions obtained by Picard's iterative scheme, DM, LM, PIA, and HPM with the exact solution, which shows that Picard's solution and exact solution make an excellent agreement while solutions derived by other mentioned techniques diverge from exact solution. Figure 3(d) shows that Picard's iterative scheme has least ||Error|| as compared to other mentioned techniques.

7.3. **Results of Example 3.** The Error estimated by Picard's iterative scheme is enlisted in Table 5 which shows that $||\text{Error}|| = \max \text{ Error } = 0.022414$. The ||Error|| estimated by Picard's method, B Spline method [19], HPM (Homotopy Perturbation method) [13], PIA (Perturbation Iteration algorithm) [14], and VIM (Variational Iteration method) [17] are listed in Table 6 which shows that

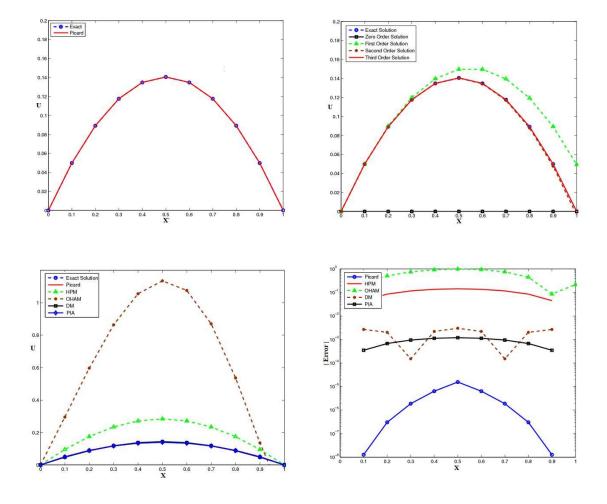


Figure 2: Graphical Illustration of Example 1

Picard's method has smallest ||Error|| amongst these methods.

Figure 4(a,b) illustrates the comparison of exact solution with approximate solutions obtained by Picard's iterative scheme. Figure 4(a) shows that both approximate and exact solutions match with each other. Figure 4(b) shows that approximate solutions of zero to fourth orders are converging to exact solution. Figure 4(c) demonstrates the comparison of approximate solutions computed by Picard's iterative scheme, BSM, HPM, PIA, and VIM with exact solution, which shows that Picard's solution and exact solution make an excellent agreement while solutions computed by other mentioned methods move away from exact solution. Figure 4(d) shows that Picard's iterative scheme has smallest ||Error|| as compared to other mentioned schemes.

Table 3: Error Estimation of Example 2

X	Exact Solution	Picard's Solution	Error
0.00	0.0000000000	0.000000000	0.0000000000
0.10	0.1144107440	0.1144110226	7.7860×10^{-7}
0.20	0.2064191156	0.2064266804	7.5640×10^{-6}
0.30	0.2738793116	0.2739270572	4.7745 × 10
0.40	0.3150893646	0.3152541108	1.6474×10^{-4}
0.50	0.3289524214	0.3293601056	8.2169×10^{-4}
0.60	0.3150893646	0.3152541108	1.6474×10^{-4}
0.70	0.2738793116	0.2739270572	4.7745 × 10
0.80	0.2064191156	0.2064266804	7.5640×10^{-6}
0.90	0.1144107440	0.1144110226	7.7860×10^{-7}
1.00	0.0000000000	0.000000000	0.0000000000

Table 4: Comparison of ||Error|| of Example 2

Method	Error
Picard's method	0.0008216900
Laplace method	0.0123778084
Decomposition method	0.0146751156
Perturbation Iteration Algorithm	0.0523780000
Homotopy Perturbation method	0.2559341111

Table 5: Error Estimation of Example 3

X	Exact Solution	Picard's Solution	Error
0.0	0.000000000	0.000000000	0.000000000
0.1	0.395805699	0.3957501818	5.5517 × 10
0.2	0.739097410	0.7385461713	5.5124×10^{-7}
0.3	1.008758260	1.0074268710	1.3314×10^{-3}
0.4	1.182536660	1.1817929190	7.4374×10^{-7}
0.5	1.242742690	1.2393843260	3.3584×10^{-3}
0.6	1.182536660	1.1712949450	1.1242 × 10
0.7	1.008758260	0.9890955265	1.9663 × 10
0.8	0.739097410	0.7166833621	2.2414×10^{-2}
0.9	0.395805699	0.3834243783	1.2381 × 10
1.0	0.000000000	0.000000000	0.000000000

8. Conclusion

Current investigation is devoted to the derivation of approximate solutions of BTEs using Picard's iterative scheme which is a reliable, effective, and efficient method for solutions of such type of equations.

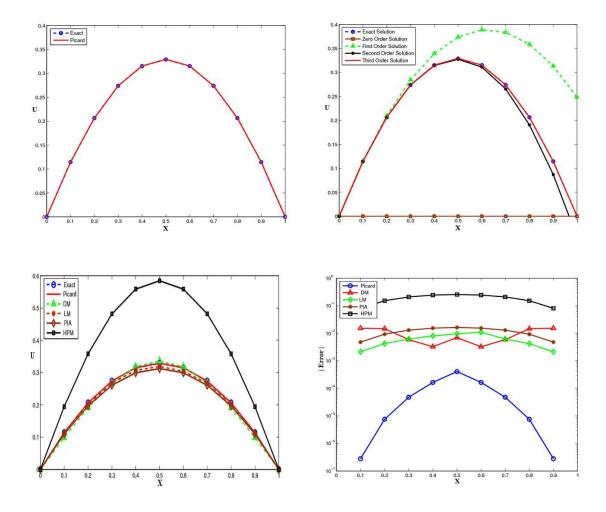


Figure 3: Graphical Illustration of Example 2

Table 6: Comparison of ||Error|| of Example 3

Method	Error
Picard's method	0.0224140000
B Spline method	0.1347528700
Homotopy Perturbation method	0.4964351282
Perturbation Iteration Algorithm	0.7001783219
Variational Iteration method	1.3943488610

Existence and uniqueness of solutions have been confirmed. Convergence and stability of solutions of numerical examples have also been examined. Furthermore, the absolute error for each numerical example has been computed. The Picard's results are compared with the results attained by DM, PIA,

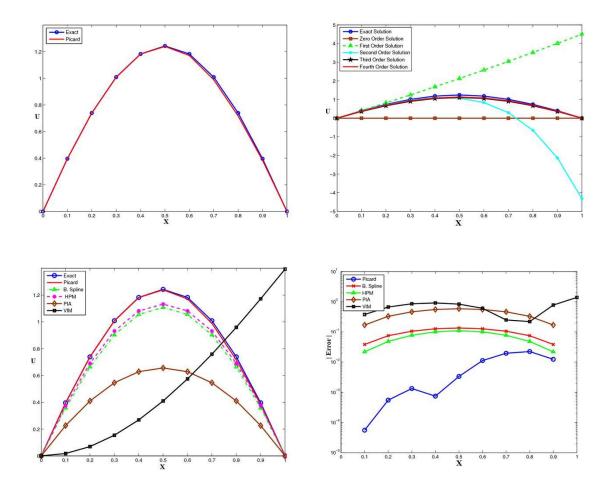


Figure 4: Graphical Illustration of Example 3

HPM, OHAM, LM, VIM, PM, and BSM through tables and graphical illustrations, and then discussed thoroughly. The derived results categorically indicate the following findings:

- · The solutions of BTEs are convergent.
- · The BTEs are Ulam-Hyers-Rassias stable.
- Picard's solutions have least || Error || amongst other mentioned techniques.
- · Picard's solutions are established to be in excellent agreement with exact solutions.

The derived results demonstrate the ability of Picard's iterative scheme in contrast of other stated methods and certainly verify the efficacy and precision of Picard's method for solution of BTEs. Moreover, these results will establish a foundation for further exploration concerning future development in differential equations and other associated fields of research. The current research investigation can be extended for the case of fractional Bratu differential equations comprising different forms of fractional derivatives.

9. **DECLARATIONS**

Conflict of Interest: The authors declare that they have no conflict of interest in the publication of this article.

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