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Analytics using Homotopy Analysis Method for Kind-II Double Dimension Fuzzy Fredholm Integral Equations

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ABSTRACT

In this manuscript, a semi-analytical method known as Homotopy Analysis Method (HAM) is proposed to solve the mathematical models that can be used to approximate the solutions. The method includes an auxiliary parameter that is an efficient way to examine the accuracy of the problems. The main purpose of this study is to explore the approximate solutions of Kind-II Double Dimension Fuzzy Fredholm Integral Equations (KII-DD-FF-IEqs) using HAM. It gives a reliable approach to confirm the convergence of the approximation series. A convergence of HAM is also presented graphically. The performance of the HAM as compared to the Homotopy Perturbation Method (HPM) is illustrated with numerical examples by solving KII-DD-FF-IEqs. The solutions sought by HAM exhibit that the proposed method is easy to apply and computationally quite attractive.

Keywords: absolute error, deformation, double dimension, fuzzy, homotopy, integral equations, maclaurin

Highlights

- The homotopy analysis method solves Kind-II Double Dimension Fuzzy Fredholm Integral Equations (KII-DD-FF-IEqs).
- Convergence of the solution series is examined using an auxiliary parameter and plots.
- Numerical results compare HAM and HPM using approximate solutions and absolute errors.

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1. INTRODUCTION

Although analytical and numerical techniques have been used for one and two-dimensional fuzzy integral equations (FIEqs), there remain some areas for semi-analytical methods to be used in two-dimensional domains.

Abel, an Italian mathematician, first introduced an integral equation (IEq) in 1825. He gave the concept of obtaining a curve equation of the considered problem. Du Bois-Reymond was the first to name this type of equation as integral equation in 1888. Various problems of science and engineering offer/require different types of IEqs. Mathematical models, such as water waves, problems of diffraction, conformal mapping and diffusing in quantum mechanics, potential theory, electrostatic and electromagnetic scattering problems, low frequency electromagnetic problems, propagation of elastical and acoustical waves, Volterra's population growth, engendering of stocked fish in new lake, biological species living together, heat transfer and heat radiation, and electric circuits and chemical kinetics have contributed in the modelling of IEqs Rahman [1], Wazwaz [2-3]. In pure and applied sense, a large number of initial boundary value (IBVe) problems supported to/supporting differential equations (DEqs) can be converted into IEqs.

In many scientific applications, Fredholm IEqs exist/apply. Erik Ivar Fredholm was a Swedish mathematician who established spectral and operator theory of IEqs Wazwaz [3]. The evolution of FIEqs, particularly fuzzy control, has a significant impact in several fields, such as physical problems, electrical engineering, mechanics, automatic control study, biology, and medicine. Fuzzy analysis components, such as fuzzy DEqs and FIEqs, play a vital role in numerical studies and have applications in different fields.

A modified homotopy perturbation method (MHPM) for the second kind of linear two-dimensional fuzzy Fredholm IEq was developed by Rivaz and Yousif [4]. Their method uses MHPM to generate approximate solutions (APx-S), transforms the original fuzzy problem into a system of crisp Fredholm IEqs, and parametrically describes the fuzzy functions. Their numerical studies, which include cases where exact solutions are obtained, show how applicable this method is.

Using the HAM, the present study explores an alternative semi-analytic framework for KII-DD-FF-IEqs. The formulation used here incorporates an auxiliary convergence control parameter, which offers a way to check and modify the convergence behaviour of the resulting approximate series, in contrast to the MHPM-based technique in Rivaz and Yousif [4]. Successive deformation terms are used to generate APx-S and endpoint wise error measures, while graphical comparisons are used to evaluate the approximation of the final sequence as compared to the exact solution.

This study makes three distinct contributions. For the FIEqs problem under consideration, it first develops a series based on HAM. Secondly, it investigates how the convergence-control parameter affects approximation and employs convergence plots to help choose an approximate parameter value. Thirdly, by comparing the computed endpoint function with the exact solution and giving their errors, it evaluates the numerical accuracy of a 10^{th} order approximation.

For fuzzy integral equations, a number of numerical and semi-analytical methods have been investigated, including iterative techniques, direct quadrature schemes, the homotopy perturbation method (HPM) by He [5], and the modified HPM. However, these methods often have problems with computational complexity, error propagation in multidimensional domains, or lack of systematic convergence control. Traditional HPM, a special case of HAM, lacks the flexibility to change convergence regions, while standard perturbation methods are ineffective in dealing with situations without small parameters. The main contribution of this work is the effective extension and application of the HAM to KII-DD-FF-IEqs. HAM provides a clear framework to control and optimize the convergence region of the solution series, leading to high computational efficiency and high accuracy. It does this by introducing a non-zero auxiliary parameter h, as demonstrated by h-curves, and a flexible auxiliary linear operator.

Ahsan et al. [6] dealt with the solution of the system of 2D-FF-IEqs, which depends upon the parametric form (PFm) of fuzzy number (FzNm), using an efficient algorithm called optimal homotopy asymptotic method. Darabi *et al*. [7] introduced a novel method to solve 2D-FF-IEqs of the second kind. Georgieva [8] solved two-dimensional nonlinear fuzzy

Volterra IEqs by using HAM. Noeiaghdam and Micula [9] applied the Lagrange–collocation method with discontinuous kernel to solve the problem of the second kind Volterra IEqs. Shiri et al. [10] discussed the existence and uniqueness of a solution with an approach which uses the Clenshaw-Curtis representation and transforms fuzzy Fredholm IEqs to a system of algebraic equations. Georgieva and Naydenova [11] obtained an APx-S of the Volterra-Fredholm FIEq. Guan et al. [12] solved Fredholm IEqs using deep learning. The numerical APx-S of linear 2D-Fredholm IEqs of the second kind via Bernstein operators were considered by Iyikal [13]. Rahby et al. [14] discussed the existence and uniqueness of the APx-S of the second kind nonlinear Volterra-Fredholm IEqs for different phenomena using Picard's method. They used Banach fixed point theorem to prove the solvability of this kind of equations. Arafa and Ramadan [15]) presented Bernoulli wavelet method for numerical APx-S of the linear system of Fredholm IEq of the second kind. Ansari et al. [16] discussed the application of the direct computation method to solve a general Fredholm integro DEqs. Ma and Li [17] presented a novel method to solve second kind Fredholm IEqs by combining L1 space discretization techniques with physics-informed neural networks. Yassin *et al*. [18] modified approximate techniques, fuzzy Fredholm HAM, to solve second kind fuzzy Fredholm system integro DEqs subject to IBVe problems. Ramadan and Osheba [19] solved the linear Fredholm IEqs of the second kind by utilizing Bernoulli polynomials combined with improved block-pulse functions. Younas and Al-Havani [20] solved the linear fuzzy system of the Volterra-Fredholm IEqs using HPM. The Banach contraction fixed point was used to demonstrate the convergence for APx-S. The symmetric fuzzification solution was obtained by using convex symmetrical triangular FzNm. Youssef *et al*. [21] proposed an iterative method for APx-S of two-dimensional weakly singular Fredholm IEq by incorporating Picard algorithm, product integration technique, and an appropriate cubature formula. Using the PFm of an intuitionistic FzNm, the nonlinear Fredholm IEq (NFIEq) was decomposed into a set of four equations by Khan et al. [22]. This set of equations was then named as the intuitionistic fuzzy NFIEq. For an exact solution to the intuitionistic fuzzy NFIEq, they used the direct computational approach.

2. PREREQUISITE TERMS FOR FUZZY IEQS

Here, we review several necessary terms and related results about

fuzzy IEqs.

Definition 1. (See, Goetschel and Voxman [23] A mapping of FzNm $\delta: R \to [0,1]$ has the conditions given below as

(a) Over R, δ is supper semi continuous.

(b) $\delta(z) = 0$, outside of $[\beta_1, \beta_2] \subset R$.

(c) For β_1, $\beta_2 \in R$ such that $\beta_3 \leq \beta_1 \leq \beta_2 \leq \beta_4$ then δ on $[\beta_3, \beta_1]$ is monotonic increasing and on $[\beta_2, \beta_4]$ monotonic decreasing.

(d) For all $z \in [\beta_1, \beta_2]$, $\delta(z) = 1$.

(e) For $\lambda \in [0,1]$, δ as fuzzy convex set, and $z_1, z_2 \in R$ then $\delta(\lambda z_1 + (1 - \lambda)z_2) \geq min\{\delta(z_1), \delta(z_2)\}$.

Since R_F is the set of all FzNms and $R \subset R_F$, then every $b \in R$ is a FzNm as $b = \chi_{\{b\}}$.

Definition 2. (See, Wu and Gong [24] For $s \in [0,1]$, the PFm of FzNm is an ordered pair function $(\underline{\delta}(s), \bar{\delta}(s))$ that satisfies conditions as

(a) Over $[0,1]$, the function $\underline{\delta}(s)$ is bounded, left continuous, and nondecreasing.

(b) Over $[0,1]$, the function $\bar{\delta}(s)$ is bounded, left continuous, and nonincreasing.

(c) $\underline{\delta}(s) \leq \bar{\delta}(s), 0 \leq s \leq 1$.

Also, for all FzNms $\delta = (\underline{\delta}(s), \bar{\delta}(s)), \Delta = (\underline{\Delta}(s), \bar{\Delta}(s))$, and $k \in$ R, the operations of addition and multiplications are as follows:

(a) $\delta = \Delta \Leftrightarrow \underline{\delta}(s) = \underline{\Delta}(s)$ and $\bar{\delta}(s) = \bar{\Delta}(s)$.

(b) (i) $\delta + \Delta = (\underline{\delta}(s) + \underline{\Delta}(s), \bar{\delta}(s) + \bar{\Delta}(s))$,

(ii) $\delta - \Delta = (\underline{\delta}(s) - \underline{\Delta}(s), \bar{\delta}(s) - \bar{\Delta}(s))$.

(c) $\delta.\Delta = \begin{cases} \underline{\delta}.\underline{\Delta}(s) = \max\{\underline{\delta}(s)\underline{\Delta}(s), \underline{\delta}(s)\bar{\Delta}(s), \bar{\delta}(s)\underline{\Delta}(s), \bar{\delta}(s)\bar{\Delta}(s)\}, \\ \bar{\delta}.\bar{\Delta}(s) = \min\{\underline{\delta}(s)\underline{\Delta}(s), \underline{\delta}(s)\bar{\Delta}(s), \bar{\delta}(s)\underline{\Delta}(s), \bar{\delta}(s)\bar{\Delta}(s)\}. \end{cases}$

(d) $k.\delta = \begin{cases} (k\underline{\delta}(s), k\bar{\delta}(s)) & \text{if } k \geq 0, \\ (k\bar{\delta}(s), k\underline{\delta}(s)) & \text{if } k < 0. \end{cases}$

Definition 3. (See, Gal [25], Kaleva [26] and, Puri and Ralescu [27]). For all FzNms $\delta = (\underline{\delta}(s), \bar{\delta}(s))$ and $\Delta = (\underline{\Delta}(s), \bar{\Delta}(s))$, the quantity $D(\delta, \Delta)$ is the distance between δ and Δ, where

$$D(\delta, \Delta) = \max\left\{\sup_{s\in[0,1]} \left|\underline{\delta}(s) - \underline{\Delta}(s)\right|, \sup_{s\in[0,1]} \left|\bar{\delta}(s) - \bar{\Delta}(s)\right|\right\}.$$

Here, (R_F, D) is a complete metric space (CMSp) (See, Puri M L, Ralescu D [28]) that possesses the properties as

(a) $\forall\, \delta, \Delta, \Pi \in R_F; D(\delta + \Pi, \Delta + \Pi) = D(\delta, \Delta)$,

(b) $\forall\, k \in R, \delta, \Delta \in R_F; D(k\delta, k\Delta) = |k| D(\delta, \Delta)$,

(c) $\forall\, \delta, \Delta, \Pi, \Lambda \in R_F; D(\Delta + \delta, \Lambda + \Pi) \leq D(\Delta, \Lambda) + D(\delta, \Pi)$,

(d) $\forall\, \tilde{0}, \delta, \Delta \in R_F; D(\Delta + \delta, \tilde{0}) \leq D(\Delta, \tilde{0}) + D(\delta, \tilde{0})$.

Definition 4. (See, Anastassiou [29]) Let $g_1, g_2 : [\beta_1, \beta_2] \to R_F$ are fuzzy real-valued functions (FRVFs). The uniform distance (UD) between g_1 and g_2 is defined as $D_u = \sup\{D(g_1(z), g_2(z)) \mid z \in [\beta_1, \beta_2]\}$.

Definition 5. (See, Wu [30]) For every $\varepsilon > 0$, there exists $d > 0$ such that $D\big(g(z), g(z_0)\big) < \varepsilon$ whenever $z \in [\beta_1, \beta_2]$, $|z - z_0| < d$, then the FRVF $g : [\beta_1, \beta_2] \to R_F$ is continuous at point $z_0 \in [\beta_1, \beta_2]$. If g is continuous at each $z_0 \in [\beta_1, \beta_2]$, then g is called a fuzzy continuous function (FCF).

Here, $C_F([\beta_1, \beta_2] \times [\beta_3, \beta_4])$ denotes the space of all FCFs $g : [\beta_1, \beta_2] \times [\beta_3, \beta_4] \to R_F$.

Definition 6. (See, Sadatrasoul and Ezzati [31]) Let $g : [\beta_1, \beta_2] \times [\beta_3, \beta_4] \to R_F$ be bounded mapping. Then the function $h_{[\beta_1,\beta_2]\times[\beta_3,\beta_4]}(g, \cdot) : R_+ \cup 0 \to R_+$ is defined as $h_{[\beta_1,\beta_2]\times[\beta_3,\beta_4]}(g, d)$ is known as the modulus of oscillation of g on $[\beta_1, \beta_2] \times [\beta_3, \beta_4]$, where

$$h_{[\beta_1,\beta_2]\times[\beta_3,\beta_4]}(g, d) = \sup\{D(g(z_1, z_2), g(s, t)) : z_1, s \in [\beta_1, \beta_2]; z_2, t \in [\beta_3, \beta_4]; \sqrt{(z_1 - s)^2 + (z_2 - t)^2} \leq d\}.$$

Definition 7. (See, Sadatrasoul and Ezzati [31]) Let $g : [\beta_1, \beta_2] \times [\beta_3, \beta_4] \to R_F$, and $\Delta_{z_1}^n : \beta_1 = z_{10} < z_{11} < \cdots < z_{1n} = \beta_2$ and $\Delta_{z_2}^n : \beta_3 = z_{20} < z_{21} < \cdots < z_{2n} = \beta_4$ are two partitions of $[\beta_1, \beta_2]$ and $[\beta_3, \beta_4]$, respectively. Considering, any points $\xi_i \in [z_{1(i-1)}, z_{1i}], \eta_j \in [z_{2(j-1)}, z_{2j}]$

for $i, j = 1, \cdots, n$ and $d: [\beta_1, \beta_2] \to R_F, \sigma: [\beta_3, \beta_4] \to R_F$.

The divisions $P_{z_1} = \left([z_{1(i-1)}, z_{1i}], \xi_j\right)$ and $P_{z_2} = \left([z_{2(j-1)}, z_{2j}], \eta_j\right)$ are denoted as $P_{z_1} = (\Delta^n, \xi)$ and $P_{z_2} = (\Delta^n, \eta)$ which are known as d-fine and σ-fine, respectively, if $[z_{1(i-1)}, z_{1i}] \subset \left(\xi_i - d(\xi_j), \xi_i + d(\xi_i)\right)$ and $[z_{2(j-1)}, z_{2j}] \subset \left(\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j)\right)$.

The function g is said to be double dimension Henstock integrable (DDHI) to $I \in R_F$ if for each $\varepsilon > 0$, there exists functions $d: [\beta_1, \beta_2] \to R_+$ and $\sigma: [\beta_3, \beta_4] \to R_+$ such that for any divisions, d-fine and σ-fine, we have

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(z_{1i} - z_{1(i-1)}\right)\left(z_{2j} - z_{2(j-1)}\right)g\left(\xi_i, \eta_j\right), \mathrm{I}\right) < \varepsilon,$$

where $\sum$ denotes the fuzzy summation. Then, I is known as the DDHI of g and is denoted as

$$I(g) = \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} g(z_1, z_2)\mathrm{d}z_1\, \mathrm{d}z_2.$$

If functions d and σ are constant, then $I(g)$ is called the Riemann integral (RmI). Here, $I \in R_F$ is called the double dimension integral of g on $[\beta_1, \beta_2] \times [\beta_3, \beta_4]$.

Definition 8. The definite integral of $g(z_1, z_2)$ over $[\beta_1, \beta_2] \times [\beta_3, \beta_4]$ is

$$\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} g(z_1, z_2)\mathrm{d}z_1\, \mathrm{d}z_2 = \mathrm{Lim}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(z_{1i} - z_{1(i-1)}\right)\left(z_{2j} - z_{2(j-1)}\right)g\left(\xi_i, \eta_j\right)\right\}.$$

if limit exists in metric D and $\left(\max\left|z_{1i} - z_{1(i-1)}\right|, \max\left|z_{2j} - z_{2(j-1)}\right|\right) \to (0,0)$.

Definition 9. If function $g(z_1, z_2)$ is continuous in metric D, then definite integrals by Goetschel and Voxman [3] are

$$\underline{\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} g(z_1,z_2,s)\mathrm{d}z_1\mathrm{d}z_2} = \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} \underline{g}(z_1,z_2,s)\mathrm{d}z_1\,\mathrm{d}z_2,$$

$$\overline{\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} g(z_1,z_2,s)\mathrm{d}z_1\mathrm{d}z_2} = \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} \bar{g}(z_1,z_2,s)\mathrm{d}z_1\,\mathrm{d}z_2.$$

3. KIND-II DOUBLE DIMENSION FUZZY FREDHOLM INTEGRAL EQUATIONS (KII-DD-FF-IEQS)

The contribution of the theory of FIEqs includes the solution of fuzzy IBVe problems. The study of elliptic type fuzzy DEqs leads to fuzzy Fredholm IEqs. The integral equation of the form given below is the nonlinear KII-DD-FF-IEq,

$$\delta(z_1,z_2) = g(z_1,z_2) + \lambda\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)H\big(\delta(s,t)\big)\mathrm{d}s\ \mathrm{d}t, (z_1,z_2) \in [\beta_1,\beta_2]\times[\beta_3,\beta_4] \quad (1)$$

Here, we consider $H(\delta(s,t)) = \delta(s,t)$ that is the equation (1) is called linear KII-DD-FF-IEq.

$$\delta(z_1,z_2) = g(z_1,z_2) + \lambda\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\delta(s,t)\mathrm{d}s\ \mathrm{d}t, (z_1,z_2) \in [\beta_1,\beta_2]\times[\beta_3,\beta_4] \quad (2)$$

In equation (2), $\lambda > 0$ and $\delta(z_1,z_2), g(z_1,z_2)$ over $[\beta_1,\beta_2]\times[\beta_3,\beta_4]$ are fuzzy functions with δ as unknown function and $k(z_1,z_2,s,t)$ over $[\beta_1,\beta_2]\times[\beta_3,\beta_4]\times[\beta_1,\beta_2]\times[\beta_3,\beta_4]$ is an arbitrary kernel function.

Now, with the help of Definition (2), we write KII-DD-FF-IEq in PFm. Let $\left(\underline{g}(z_1,z_2,r), \bar{g}(z_1,z_2,r)\right)$ and $\left(\underline{\delta}(z_1,z_2,r), \bar{\delta}(z_1,z_2,r)\right), r \in [0,1]$ and $(z_1,z_2) \in [\beta_1,\beta_2]\times[\beta_3,\beta_4]$, be PFm of $g(z_1,z_2)$ and $\delta(z_1,z_2)$, respectively. Then, PFm of KII-DD-FF-IEq is as follows:

$$\left.\begin{aligned}\underline{\delta}(z_1,z_2,r) &= \underline{g}(z_1,z_2,r) + \lambda\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} \omega_1\big(z_1,z_2,s,t,\underline{\delta}(s,t,r),\bar{\delta}(s,t,r)\big)\mathrm{d}s\ \mathrm{d}t\\ \bar{\delta}(z_1,z_2,r) &= \bar{g}(z_1,z_2,r) + \lambda\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} \omega_2\big(z_1,z_2,s,t,\underline{\delta}(s,t,r),\bar{\delta}(s,t,r)\big)\mathrm{d}s\ \mathrm{d}t\end{aligned}\right\} \quad (3)$$

where

$$\omega_1\left(z_1, z_2, s, t, \underline{\delta}(s,t,r), \bar{\delta}(s,t,r)\right) = \begin{cases} k(z_1, z_2, s, t)\underline{\delta}(s,t,r); k(z_1, z_2, s, t) \geq 0 \\ k(z_1, z_2, s, t)\bar{\delta}(s,t,r); k(z_1, z_2, s, t) < 0 \end{cases}$$

and

$$\omega_2\left(z_1, z_2, s, t, \underline{\delta}(s,t,r), \bar{\delta}(s,t,r)\right) = \begin{cases} k(z_1, z_2, s, t)\bar{\delta}(s,t,r); k(z_1, z_2, s, t) \geq 0 \\ k(z_1, z_2, s, t)\underline{\delta}(s,t,r); k(z_1, z_2, s, t) < 0 \end{cases}$$

for each $r \in [0,1], z_1 \in [\beta_1, \beta_2]$ and $z_2 \in [\beta_3, \beta_4]$. Equation (3) is a system of linear kind-II Fredholm IEqs with three variables in crisp form.

4. HOMOTOPY

Homotopy is traced back to French mathematician Jules Henri Poincaré (1854-1912). In mathematics, it describes deformation or type of continuous variation. It also defines an association among different things containing the same characteristics in some aspects. For example, the state of a cup of coffee (or circle) can be ceaselessly disfigured into a state of doughnut (or square/an ellipse) but it cannot be distorted persistently into a state of football.

Consider a family of two different real functions (RFns), $8z(z-1)$ and $\sin(\pi z)$, is

$$\mathbb{H}(z; q) = q[8z(z-1)] + (1-q)\sin(\pi z),$$

where $z \in [0, 1]$ is the independent variable (ID-Vr) and $q \in [0, 1]$ is the embedding parameter (EBgPr). Then, a new real function $\mathbb{H}(z; q)$ depends on z as well as q.

For $q = 0$: $\mathbb{H}(z; 0) = \sin(\pi z)$; and for $q = 1$: $\mathbb{H}(z; 1) = 8z(z-1)$; $z \in [0, 1]$.

As q moves 0 to 1, the new function constantly changes from trigonometric to polynomial function. In topology, $8z(z-1)$ and $\sin(\pi z)$ are homotopic, $\mathbb{H}(z; q)$ is homotopy, signified as

$$\mathbb{H}: \sin(\pi z) \sim 8z(z-1).$$

Definition 10. Let $g_1, g_2 \in C[\beta_1, \beta_2]$, where C be set of all continuous RFns (CRFns). In general, if g_1 can be deformed continuously into g_2, one can construct a homotopy

$\mathbb{H}: g_1 \sim g_2$

In this sense

$$\mathbb{H}(z;q) = qg_1(z) + (1-q)g_2(z);\ z \in [\beta_1, \beta_2].$$

Although, a CRFn, sin(z), cannot be changed persistently into a discontinuous function, step function, $s(z)$

$$s(z) = \begin{cases} 1 & \text{if} \quad z < 0, \\ 0 & \text{if} \quad z = 0, \\ -1 & \text{if} \quad z > 0. \end{cases}$$

Definition 11. Consider $g_1(z), g_2(z)$ and $\mathbb{H}(z;q)$ be continuous functions, Z_1 and Z_2 be topological spaces. A homotopy between g_1 and g_2 from Z_1 to Z_2 is to be $\mathbb{H}: Z_1 \times [0,1] \to Z_2$ from the product of Z_1 and $[0,1]$ to Z_2 in order that, provided $z \in Z_1$, then $\mathbb{H}(z;0) = g_1(z)$ and $\mathbb{H}(z;1) = g_2(z)$.

Definition 12. In a homotopy of equations (HmEqs) or functions, the embedding parameter $q \in [0,1]$ is known as homotopy-parameter (HmPr).

Definition 13. Consider original equation E_1, with at least one unknown solution δ and E_0 is an initial equation with known solution δ_0. In the event one can build a HmEqs, $\tilde{E}(q): E_0 \sim E_1$ to such an extent that, as q moves 0 to 1, $\tilde{E}(q)$ varies or deforms ceaselessly from E_0 to E_1, while its solutions varies continuously from δ_0 to δ, at that point this sort of HmEqs is known as the zero-order deformation equation (ZODEq).

Definition 14. Consider t be temporal and z be spatial ID-Vrs, q a HMPr and $\tilde{E}(q)$ is ZODEq. Given nonlinear original equation E_1, with at least one solution $\delta(z,t)$ and E_0, an underlying equation, with known introductory solution $\delta_0(z,t)$. Let $\tilde{E}(q)$ connects E_1 and E_0. Assuming that $\tilde{E}(q)$ is legitimately built that its answer $\phi(z,t;q)$ exists and at $q = 0$ is analytic, we have maclaurin homotopy series (MHMS)

$$\phi(z,t;q) \sim \delta_0(z,t) + \sum_{n=1}^{+\infty} \delta_n(z,t)q^n, q \in [0,1].$$

and the homotopy series (HmS)

$$\phi(z,t;1) \sim \delta_0(z,t) + \sum_{n=1}^{+\infty} \delta_n(z,t).$$

The equations in which unknown, $\delta_n(z,t)$, involved are known as N-order deformation equation (NODEq).

Definition 15. If solution $\phi(z,t;q)$ of $\tilde{E}(q)$: $E_0 \sim E_1$ exists and about q is analytic, then HmS solution of E_1 is

$$\delta(z,t) = \delta_0(z,t) + \sum_{n=1}^{+\infty} \delta_n(z,t)$$

where $\tilde{E}(q)$ the ZODEq, E_0 initial equation, E_1 the original equation, and $q \in [0,1]$.

Then, the M-order HmS-approximation is

$$\delta(z,t) \approx \delta_0(z,t) + \sum_{n=1}^{M} \delta_n(z,t).$$

Definition 16. If $\sum_{m=1}^{\infty} \delta_m(z_1,z_2)q^m$ be an arbitrary $MHmS$, then the homotopy-derivative of $MHmS$ are as

$$\mathbb{D}_m\left\{\sum_{m=1}^{\infty} \delta_m(z_1,z_2)q^m\right\} = \delta_m(z_1,z_2), \tag{4}$$

$$\mathbb{D}_m\left\{q\sum_{m=1}^{\infty} \delta_m(z_1,z_2)q^m\right\} = \mathbb{D}_{m-1}\left\{\sum_{m=1}^{\infty} \delta_m(z_1,z_2)q^m\right\}$$

$$= \delta_{m-1}(z_1,z_2), \tag{5}$$

$$\mathbb{D}_m\{q\mathbb{N}[\phi(z_1,z_2;q)]\} = \mathbb{D}_{m-1}\{\mathbb{N}[\phi(z_1,z_2;q)]\}. \tag{6}$$

5. MODELLING USING HOMOTOPY ANALYSIS METHOD (HAM)

Consider the given equation E_1 having nonlinear operator (NLrO) $\mathbb{N}$ and ID-Vrs z_1 and z_2, written as

$$\mathbb{N}\big(\delta(z_1,z_2)\big) = 0. \tag{7}$$

At least one solution of equation (7) is unknown function $\delta(z_1,z_2)$.

5.1. Choice of Initial Approximation

Assuming the primary approximation equation E_0 of true solution

$\delta(z_1, z_2)$ be $\delta_0(z_1, z_2)$, it should satisfy the governing starting/boundary conditions and capture the key linear components of the problems without going against the limits of the metric space (R_F, D), according to the fundamental principle of HAM. We ensure that the initial guess $\delta(z_1, z_2) = \delta_0(z_1, z_2)$ naturally includes the non-homogeneous source terms of FIEqs. This greatly speeds up convergence in the subsequent deformation equations. In the following equation, we construct HmEqs $\tilde{E}(q): E_0 \sim E_1$ as q raises 0 to 1, $\tilde{E}(q)$ differs persistently E_0 to E_1, while its solution exists for q and deforms ceaselessly $\delta_0(z_1, z_2)$ to $\delta(z_1, z_2)$.

5.2. Choice of Auxiliary Linear Operator and Auxiliary Function

The homotopy is as follows:

$$(1-q)\mathbb{L}[\phi(z_1, z_2; q) - \delta_0(z_1, z_2)] - qh\mathbb{H}(z_1, z_2)\mathbb{N}[\phi(z_1, z_2; q)] = \mathbb{H}[\phi(z_1, z_2; q); \delta_0(z_1, z_2), \mathbb{H}(z_1, z_2), h, q]. \quad (8)$$

In equation (8), we choose $q \in [0, 1]$ as HmPr, $\mathbb{H}(z_1, z_2) \neq 0$ an auxiliary function (AFn), $\mathbb{L}$ the simple identity auxiliary linear operator (ALrO) with characteristic $\mathbb{L}[R(z_1, z_2)] = 0$ when $R(z_1, z_2) = 0$ because the higher-order unscaled terms of the transformed crisp parametric equations are directly equivalent to the linear operator $\mathbb{L}$. This process transforms the higher-order deformation equations into simple recurrence relations and does not involve any secular terms or unnecessary integral burden.

5.3. How to Control Convergence with Auxiliary Parameter?

Unlike traditional perturbation or standard homotopy methods, HAM gives a convenient way to control and adjust the convergence region by using the auxiliary parameter (APr), $h \neq 0$. The APr has a significant impact on the convergence rate of the HAM solution series. To determine the valid region for h, we have plotted the h-curves, which represent the value of the nth-order approximation solution against h at a fixed point. The appropriate region of h where the solution series converges is clearly indicated by the flat segments (parallel surfaces) present in these curves. Rigorous choice of h within this optimal plateau ensures that the Maclaurin homotopy series converges accurately to the exact solution.

We have freedom to select the underlying guess $\delta_0(z_1, z_2)$, the non-zero $h, \mathbb{H}(z_1, z_2)$ and $\mathbb{L}$. Now, enforcing equation (8) to be zero,

$$\mathbb{H}[\phi(z_1, z_2; q); \delta_0(z_1, z_2), \mathbb{H}(z_1, z_2), h, q] = 0, \quad (9)$$

We have, the so-called ZODEq

$$(1-q)\mathbb{L}[\phi(z_1, z_2; q) - \delta_0(z_1, z_2)] = qh\mathbb{H}(z_1, z_2)\mathbb{N}[\phi(z_1, z_2; q)]. \quad (10)$$

Since $h \neq 0$ and $\mathbb{H}(z_1, z_2) \neq 0$, so by settin $q = 0$ and $q = 1$ in equation (10), we obtain

$$\phi(z_1, z_2; 0) = \delta_0(z_1, z_2), \quad (11)$$

and

$$\phi(z_1, z_2; 1) = \delta(z_1, z_2). \quad (12)$$

Equations (11) and (12) describe deformation in homotopy of Liao [32], that is, q moves 0 to 1, $\phi(z_1, z_2; q)$ deforms continuously from $\delta_0(z_1, z_2)$ to $\delta(z_1, z_2)$. When $\phi(z_1, z_2; q)$ follows to Taylors theorem, we can write it in a power series of q as

$$\phi(z_1, z_2; q) = \delta_0(z_1, z_2) + \sum_{m=1}^{\infty} \delta_m(z_1, z_2) q^m, \quad (13)$$

where

$$\delta_m(z_1, z_2) = \frac{1}{m!}\left[\frac{\partial^m \phi(z_1, z_2; q)}{\partial q^m}\right]\Bigg|_{q=0}. \quad (14)$$

Considering initial guess $\delta_0(z_1, z_2)$, nonzero $h, \mathbb{L}$ and $\mathbb{H}(z_1, z_2)$ be so rightly chosen, so that equation (13) converges at $q = 1$. Then, under the above consideration, equation (13) becomes

$$\delta(z_1, z_2) = \phi(z_1, z_2; 1) = \delta_0(z_1, z_2) + \sum_{m=1}^{\infty} \delta_m(z_1, z_2). \quad (15)$$

Using equation (13), we can write equation (10) as

$$\mathbb{L}\left[\sum_{m=1}^{\infty} \delta_m(z_1, z_2) q^m\right] - q\mathbb{L}\left[\sum_{m=1}^{\infty} \delta_m(z_1, z_2) q^m\right] = qh\mathbb{H}(z_1, z_2)\mathbb{N}[\phi(z_1, z_2; q)]. \quad (16)$$

Differentiating "m" times the equation (16) w r t "q" we obtain

$$\mathbb{L}[\mathbb{D}_m\{\textstyle\sum_{m=1}^{\infty} \delta_m(z_1, z_2) q^m\}] - \mathbb{L}[\mathbb{D}_m\{q \textstyle\sum_{m=1}^{\infty} \delta_m(z_1, z_2) q^m\}] =$$

$$h\mathbb{H}(z_1, z_2)\mathbb{D}_m\{q\mathbb{N}[\phi(z_1, z_2; q)]\}. \quad (17)$$

Using equations (4), (5) and (6), we obtain equation (18) called M-order deformation equation (MODEq) as follows:

$$m!\,\mathbb{L}[\delta_m(z_1, z_2) - \delta_{m-1}(z_1, z_2)] = mh\mathbb{H}(z_1, z_2)\frac{\partial^{m-1}\mathbb{N}[\phi(z_1, z_2; q)]}{\partial q^{m-1}}\bigg|_{q=0} \quad (18)$$

Assume $\phi(z_1, z_2; q) = \sum_{m=0}^{\infty} \delta_m(z_1, z_2)q^m$ be MHmS, $\mathbb{L}$ an ALrO with $\mathbb{L}(0) = 0$ and independent of $q \in [0, 1]$, where q be the HmPr . It holds

$$\mathbb{D}_m\{(1-q)\mathbb{L}[\phi(z_1, z_2; q) - \delta_0(z_1, z_2)]\} = \mathbb{L}[\delta_m(z_1, z_2) - \chi_m\delta_{m-1}(z_1, z_2)], \quad (19)$$

where

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \quad (20)$$

Therefore, using equation (19), equation (18) becomes

$$\mathbb{L}[\delta_m(z_1, z_2) - \chi_m\delta_{m-1}(z_1, z_2)] = h\mathbb{H}(z_1, z_2)\mathbb{R}_m\big(\delta_{m-1}(z_1, z_2)\big), \quad (21)$$

where

$$\mathbb{R}_m\big(\delta_{m-1}(z_1, z_2)\big) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbb{N}[\phi(z_1, z_2; q)]}{\partial q^{m-1}}\bigg|_{q=0}.$$

6. HAM FOR SOLVING KII-DD-FF-IEq

Consider $\lambda = 1$ and $k(z_1, z_2, s, t) \geq 0$ over $[\beta_1, \beta_2] \times [\beta_3, \beta_4] \times [\beta_1, \beta_2] \times [\beta_3, \beta_4]$, so PFm of equation (2) is

$$\underline{\delta}(z_1, z_2, r) = \underline{g}(z_1, z_2, r) + \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1, z_2, s, t)\underline{\delta}(s, t, r)\mathrm{d}s\,\mathrm{d}t, \quad (22)$$

$$\bar{\delta}(z_1, z_2, r) = \bar{g}(z_1, z_2, r) + \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1, z_2, s, t)\bar{\delta}(s, t, r)\mathrm{d}s\,\mathrm{d}t. \quad (23)$$

To find APx-S of equations (22) and (23) with the help of HAM, first we rewrite equation (22) as

$$\mathbb{N}\left(\underline{\delta}(z_1,z_2,r)\right) = \underline{\delta}(z_1,z_2,r) - \underline{g}(z_1,z_2,r) - \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\underline{\delta}(s,t,r)\mathrm{d}s\,\mathrm{d}t. \quad (24)$$

According to equations (15) and (21), we have

$$\underline{\delta}(z_1,z_2,r) = \underline{\delta_0}(z_1,z_2,r) + \sum_{m=1}^{\infty} \underline{\delta_m}(z_1,z_2,r), \quad (25)$$

$$\mathbb{L}\left[\underline{\delta_m}(z_1,z_2,r) - \chi_m\underline{\delta_{m-1}}(z_1,z_2,r)\right] = h\mathbb{H}(z_1,z_2)\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right), \quad (26)$$

and we define $\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right)$ as

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbb{N}[\phi(z_1,z_2;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

we use equation (13) as

$$\phi(z_1,z_2;q) = \sum_{m=0}^{\infty} \underline{\delta_m}(z_1,z_2,r)q^m,$$

so,

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbb{N}\left[\sum_{m=0}^{\infty}\underline{\delta_m}(z_1,z_2,r)q^m\right]}{\partial q^{m-1}}\bigg|_{q=0}$$

$$= \frac{\partial^{m-1}\left[\begin{matrix}\sum_{m=0}^{\infty}\underline{\delta_m}(z_1,z_2,r)q^m - \underline{g}(z_1,z_2,r) - \\ \int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\sum_{m=0}^{\infty}\underline{\delta_m}(s,t,r)q^m\,\mathrm{d}s\,\mathrm{d}t\end{matrix}\right]}{(m-1)!\,\partial q^{m-1}}\bigg|_{q=0},$$

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \frac{\partial^{m-1}\left[\sum_{m=0}^{\infty}\ \underline{\delta_m}(z_1,z_2,r)q^m\right]}{(m-1)!\,\partial q^{m-1}}\Bigg|_{q=0} - \frac{\partial^{m-1}\left[\underline{g}(z_1,z_2,r)\right]}{(m-1)!\,\partial q^{m-1}}\Bigg|_{q=0}$$

$$\frac{\partial^{m-1}\left[\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\sum_{m=0}^{\infty}\ \underline{\delta_m}(s,t,r)q^m\,\mathrm{d}s\,\mathrm{d}t\right]}{(m-1)!\,\partial q^{m-1}}\Bigg|_{q=0},$$

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \mathbb{D}_{m-1}\left[\sum_{m=0}^{\infty}\ \underline{\delta_m}(z_1,z_2,r)q^m\right] - \mathbb{D}_{m-1}\left[\underline{g}(z_1,z_2,r)\right]$$

$$-\mathbb{D}_{m-1}\left[\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\sum_{m=0}^{\infty}\ \underline{\delta_m}(s,t,r)q^m\,\mathrm{d}s\,\mathrm{d}t\right],$$

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \mathbb{D}_{m-1}\left[\sum_{m=0}^{\infty}\ \underline{\delta_m}(z_1,z_2,r)q^m\right] - \mathbb{D}_{m-1}\left[\underline{g}(z_1,z_2,r)\right]$$

$$-\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\mathbb{D}_{m-1}\left[\sum_{m=0}^{\infty}\ \underline{\delta_m}(s,t,r)q^m\right]\mathrm{d}s\,\mathrm{d}t,$$

$$\mathbb{R}_m\left(\underline{\delta_{m-1}}(z_1,z_2,r)\right) = \underline{\delta_{m-1}}(z_1,z_2,r) - (1-\chi_m)\underline{g}(z_1,z_2,r) - \int_{\beta_4}^{\beta_3}\int_{\beta_1}^{\beta_2} k(z_1,z_2,s,t)\underline{\delta_{m-1}}(s,t,r)\mathrm{d}s\,\mathrm{d}t. \quad (27)$$

Putting equation (27) in equation (26), we obtain

$$\mathbb{L}\left[\underline{\delta_m}(z_1,z_2,r)-\chi_m\underline{\delta_{m-1}}(z_1,z_2,r)\right]=h\mathbb{H}(z_1,z_2)\left[\underline{\delta_{m-1}}(z_1,z_2,r)\right.$$
$$-(1$$
$$\left.-\chi_m)\underline{g}(z_1,z_2,r)-\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\underline{\delta_{m-1}}(s,t,r)\mathrm{d}s\ \mathrm{d}t\right], \quad (28)$$

We take an ALrO $\mathbb{L}[r(z_1,z_2)]=r(z_1,z_2)$, AFn $\mathbb{H}(z_1,z_2)=1$, and an initial guess $\underline{\delta_0}(z_1,z_2,r)=\underline{g}(z_1,z_2,r)$. When these and equation (20) are substituted in equation (28), we have the following recurrence relation

$$\underline{\delta_m}(z_1,z_2,r)-\chi_m\underline{\delta_{m-1}}(z_1,z_2,r)=h\left[\underline{\delta_{m-1}}(z_1,z_2,r)-(1-\right.$$
$$\left.\chi_m)\underline{g}(z_1,z_2,r)-\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\underline{\delta_{m-1}}(s,t,r)\mathrm{d}s\ \mathrm{d}t\right]$$

If $m=1,\chi_m=0$ then the above equation becomes

$$\underline{\delta_1}(z_1,z_2,r)=h\left[\underline{\delta_0}(z_1,z_2,r)-\underline{g}(z_1,z_2,r)-\right.$$
$$\left.\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\underline{\delta_0}(s,t,r)dsdt\right],$$

$$\underline{\delta_1}(z_1,z_2,r)=-h\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\underline{\delta_0}(s,t,r)dsdt. \quad (29)$$

If $m=2,3,4,\dots,m;\chi m=1$ then

$$\underline{\delta_m}(z_1,z_2,r)=(1+h)\underline{\delta_{m-1}}(z_1,z_2,r)-h\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\underline{\delta_{m-1}}(s,t,r)\mathrm{d}s\ \mathrm{d}t. \quad (30)$$

Now, for equation (23), we adopt the same procedure and obtain APx-S as

$$\overline{\delta_1}(z_1,z_2,r)=-h\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\overline{\delta_0}(s,t,r)dsdt, \quad (31)$$

For $m=2,3,4,\dots,m$, we have

$$\overline{\delta_m}(z_1,z_2,r)=(1+h)\overline{\delta_{m-1}}(z_1,z_2,r)-h\int_{\beta_3}^{\beta_4}\int_{\beta_1}^{\beta_2}k(z_1,z_2,s,t)\overline{\delta_{m-1}}(s,t,r)\mathrm{d}s\ \mathrm{d}t. \quad (32)$$

So, the approximate solution for equation (2) be

$$\delta(z_1,z_2,r)=\left(\underline{\delta}(z_1,z_2,r),\bar{\delta}(z_1,z_2,r)\right).$$

7. NUMERICAL APPLICATIONS

Now, we present numerical application to explore the performance of the HAM. We evaluate the approximate solution using $n+1$ terms.

$$\delta_{APx-S[n]}(s,t,r,h) = \left(\underline{\delta}_{APx-S[n]}(s,t,r,h), \bar{\delta}_{APx-S[n]}(s,t,r,h)\right),$$

where

$$\underline{\delta}_{APx-S[n]}(s,t,r,h) = \sum_{m=0}^{n} \underline{\delta_m}(s,t,r,h),$$

and

$$\bar{\delta}_{APx-S[n]}(s,t,r,h) = \sum_{m=0}^{n} \overline{\delta_m}(s,t,r,h).$$

The following computations are performed using the software, Maple [33].

Example 1. Consider Kind-II Double Dimension Fuzzy Fredholm Integral Equation Rivaz and Yousefi [10] as follows:

$$\delta(z_1,z_2) = g(z_1,z_2) + \int_0^1 \int_0^1 z_1^2 z_2 s\, \delta(s,t) ds\, dt, 0 \le z_1, z_2 \le 1, \tag{33}$$

with parametric form

$$\left.\begin{aligned} \underline{\delta}(z_1,z_2) &= \underline{g}(z_1,z_2) + \int_0^1 \int_0^1 z_1^2 z_2 s \underline{\delta}(s,t) ds dt, \\ \bar{\delta}(z_1,z_2) &= \bar{g}(z_1,z_2) + \int_0^1 \int_0^1 z_1^2 z_2 s \bar{\delta}(s,t) ds dt, \end{aligned}\right\} \tag{34}$$

where $k(z_1,z_2,s,t) = z_1^2 z_2 s, 0 \le s,t \le 1$ and

$$\left(g(z_1,z_2)\right)(r) = \left(z_1 \sin\left(\frac{z_2}{2}\right)(r+r^2), z_1 \sin\left(\frac{z_2}{2}\right)(4-r-r^3)\right), 0 \le r \le 1,$$

having the exact solution, by using the direct method

$$\left.\begin{aligned}\underline{\delta}(z_1,z_2,r) &= z_1\left(\sin\left(\frac{z_2}{2}\right)-\frac{16}{21}\left(\cos\left(\frac{1}{2}\right)-1\right)z_1z_2\right)(r+r^2),\\ \bar{\delta}(z_1,z_2,r) &= z_1\left(\sin\left(\frac{z_2}{2}\right)-\frac{16}{21}\left(\cos\left(\frac{1}{2}\right)-1\right)z_1z_2\right)(4-r-r^3).\end{aligned}\right\} \quad (35)$$

The algorithm of HAM is implemented for KII-DD-FF-IEq using Maple software. We obtained the following results. The terms of the HmS are

$$\underline{\delta_0}(z_1,z_2,r) = z_1\sin\left(\frac{z_2}{2}\right)(r+r^2)$$

$$\underline{\delta_1}(z_1,z_2,r) = \frac{2}{3}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)$$

$$\underline{\delta_2}(z_1,z_2,r) = \frac{1}{12}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)$$

$$\underline{\delta_3}(z_1,z_2,r) = \frac{1}{96}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^2$$

$$\underline{\delta_4}(z_1,z_2,r) = \frac{1}{768}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^3$$

$$\underline{\delta_5}(z_1,z_2,r) = \frac{1}{6144}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^4$$

$$\underline{\delta_6}(z_1,z_2,r) = \frac{1}{49152}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^5$$

$$\underline{\delta_7}(z_1,z_2,r) = \frac{1}{393216}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^6$$

$$\underline{\delta_8}(z_1,z_2,r) = \frac{1}{3145728}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^7$$

$$\underline{\delta_9}(z_1,z_2,r) = \frac{1}{25165824}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^8$$

$$\underline{\delta_{10}}(z_1,z_2,r) = \frac{1}{201326592}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)(8+7h)^9$$

and

$$\overline{\delta_0}(z_1,z_2,r) = z_1\sin\left(\frac{z_2}{2}\right)(4-r-r^3)$$

$$\overline{\delta_1}(z_1,z_2,r)=\frac{2}{3}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)$$

$$\overline{\delta_2}(z_1,z_2,r)=\frac{1}{12}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)$$

$$\overline{\delta_3}(z_1,z_2,r)=\frac{1}{96}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^2$$

$$\overline{\delta_4}(z_1,z_2,r)=\frac{1}{768}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^3$$

$$\overline{\delta_5}(z_1,z_2,r)=\frac{1}{6144}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^4$$

$$\overline{\delta_6}(z_1,z_2,r)=\frac{1}{49152}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^5$$

$$\overline{\delta_7}(z_1,z_2,r)=\frac{1}{393216}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^6$$

$$\overline{\delta_8}(z_1,z_2,r)=\frac{1}{3145728}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^7$$

$$\overline{\delta_9}(z_1,z_2,r)=\frac{1}{25165824}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^8$$

$$\overline{\delta_{10}}(z_1,z_2,r)=\frac{1}{201326592}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)(8+7h)^9$$

We approximate $\underline{\delta}(s,t,r,h)$ as

$$\underline{\delta}_{APx-S[n]}(s,t,r,h)=\sum_{m=0}^{n}\underline{\delta_m}(s,t,r,h)=z_1\sin\left(\frac{z_2}{2}\right)(r+r^2)$$

$$+\frac{2}{3}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(r+r^2)\left\{1+\sum_{m=2}^{n}\frac{1}{2^{3m-3}}(8+7h)^{m-1}\right\}$$

and $\bar{\delta}(s,t,r,h)$ as

$$\bar{\delta}_{APx-S[n]}(s,t,r,h)=\sum_{m=0}^{n}\overline{\delta_m}(s,t,r,h)=z_1\sin\left(\frac{z_2}{2}\right)(4-r-r^3)$$

$$+\frac{2}{3}hz_1^2z_2\left(\cos\left(\frac{1}{2}\right)-1\right)(4-r-r^3)\left\{1+\sum_{m=2}^{n}\frac{1}{2^{3m-3}}(8+7h)^{m-1}\right\}$$

To obtain HmS, the algorithm of HPM is applied via Maple software for solving KII-DD-FF-IEq. The results are given below.

$$\underline{\delta_0}(z_1,z_2,r)=z_1\sin\left(\frac{z_2}{2}\right)(r+r^2),$$

$$\underline{\delta_1}(z_1,z_2,r)=\frac{2}{3}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_2}(z_1,z_2,r)=\frac{1}{12}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_3}(z_1,z_2,r)=\frac{1}{96}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_4}(z_1,z_2,r)=\frac{1}{768}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_5}(z_1,z_2,r)=\frac{1}{6144}z_1^2z_2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_6}(z_1,z_2,r)=\frac{1}{49152}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_7}(z_1,z_2,r)=\frac{1}{393216}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_8}(z_1,z_2,r)=\frac{1}{3145728}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_9}(z_1,z_2,r)=\frac{1}{25165824}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2),$$

$$\underline{\delta_{10}}(z_1,z_2,r)=\frac{1}{201326592}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2).$$

and

$$\bar{\delta}_0(z_1,z_2,r) = z_1 \sin\left(\frac{z_2}{2}\right)(4-r-r^3),$$

$$\bar{\delta}_1(z_1,z_2,r) = \frac{2}{3} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_2(z_1,z_2,r) = \frac{1}{12} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_3(z_1,z_2,r) = \frac{1}{96} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_4(z_1,z_2,r) = \frac{1}{768} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_5(z_1,z_2,r) = \frac{1}{6144} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_6(z_1,z_2,r) = \frac{1}{49152} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_7(z_1,z_2,r) = \frac{1}{393216} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_8(z_1,z_2,r) = \frac{1}{3145728} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_9(z_1,z_2,r) = \frac{1}{25165824} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3),$$

$$\bar{\delta}_{10}(z_1,z_2,r) = \frac{1}{201326592} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3).$$

We approximate $\underline{\delta}(s,t,r)$ with

$$\underline{\delta}_{APx-S[n]}(s,t,r) = \sum_{m=0}^{n} \underline{\delta_m}(s,t,r) = z_1 \sin\left(\frac{z_2}{2}\right)(r+r^2)$$

$$+\frac{2}{3} z_2 z_1^2 \left(1-\cos\left(\frac{1}{2}\right)\right)(r+r^2)\left\{\sum_{m=2}^{n} \frac{1}{2^{3m-3}}+1\right\},$$

and $\bar{\delta}(s,t,r)$ with

$$\bar{\delta}_{APx-S[n]}(s,t,r)=\sum_{m=0}^{n}\overline{\delta_m}(s,t,r)=z_1\sin\left(\frac{z_2}{2}\right)(4-r-r^3)$$

$$+\frac{2}{3}z_2z_1^2\left(1-\cos\left(\frac{1}{2}\right)\right)(4-r-r^3)\left\{\sum_{m=2}^{n}\frac{1}{2^{3m-3}}+1\right\},n\geq 2.$$

8. COMPARISON

Figures 1 and 2, the 10th-order curves of h for APx-S, show the convergent regions of the solution series used by HAM $\left(\underline{\delta}_{APx-S[10]}(s,t,r,h),\bar{\delta}_{APx-S[10]}(s,t,r,h)\right)$ for Example 1. The comparison between APx-S $\left(\underline{\delta}_{APx-S[10]}(s,t,r,-1.0),\bar{\delta}_{APx-S[10]}(s,t,r,-1.0)\right)$ and exact solution $\left(\underline{\delta}(z_1,z_2,r),\bar{\delta}(z_1,z_2,r)\right)$ used by HAM with metric $D(\underline{\delta},\bar{\delta})$ using Definition 3 for Example 1 are given by Figures 3 and 4, respectively. Figure 5 also shows the graphical representation of comparison between the exact solution and APx-S used by HAM for Example 1. Table 1 and 2 describes numerical results including exact solution, approximate solution, and absolute errors used by HAM and HPM for Example 1. Here for

$$\text{maxD}\left(\underline{\delta}_{APx-S[10]}(s,t,r,-1.1),\underline{\delta}(z_1,z_2,r)\right)=3.3\times10^{-11},$$

and

$$\text{maxD}\left(\bar{\delta}_{APx-S[10]}(s,t,r,-1.1),\bar{\delta}(z_1,z_2,r)\right)=5.6\times10^{-11}.$$

Numerical comparison between APx-S $\left(\underline{\delta}_{APx-S[10]}(s,t,r),\bar{\delta}_{APx-S[10]}(s,t,r)\right)$ and exact solution $\left(\underline{\delta}(z_1,z_2,r),\bar{\delta}(z_1,z_2,r)\right)$ used by HPM with metric $D(\underline{\delta},\bar{\delta})$ using Definition 3 for Example 1, where

$$\text{maxD}\left(\underline{\delta}_{APx-S[10]}(s,t,r),\underline{\delta}(z_1,z_2,r)\right)=3.7\times10^{-11},$$

and

$$\text{maxD}\left(\bar{\delta}_{APx-S[10]}(s,t,r),\bar{\delta}(z_1,z_2,r)\right)=1.0\times10^{-10}.$$

$$\left[\frac{1}{2}\sin\left(\frac{1}{4}\right)+\frac{1}{12}h\left(\cos\left(\frac{1}{2}\right)-1\right)\left\{1+\sum_{m=2}^{n}\frac{(8+7h)^{m-1}}{2^{3m-3}}\right\}\right](r^2+r)$$

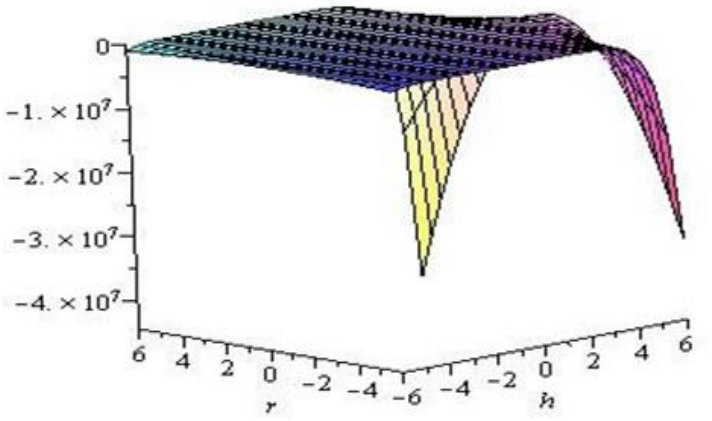


Figure 1. 10th-order Curve of h for APx-S Used by HAM $\left(\underline{\delta}_{APx-S[n]}(s,t,r,h)\right)$ for Example 1.

$$\left[\frac{1}{2}\sin\left(\frac{1}{4}\right)+\frac{1}{12}h\left(\cos\left(\frac{1}{2}\right)-1\right)\left\{1+\sum_{m=2}^{n}\frac{(8+7h)^{m-1}}{2^{3m-3}}\right\}\right](4-r^3-r)$$

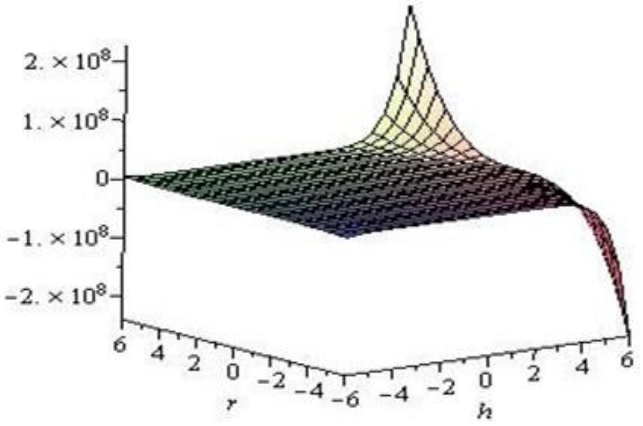


Figure 2. 10th-order Curve of h for APx-S Used by HAM $\left(\bar{\delta}_{APx-S[10]}(s,t,r,h)\right)$ for Example 1.

$$\frac{2}{3}x_1{}^2x_2\left(1-\cos\left(\frac{1}{2}\right)\right)(r^2+r)\left[\frac{8}{7}+h\left[1+\sum_{m=2}^{n}\frac{(8+7h)^{m-1}}{2^{3m-3}}\right]\right]$$

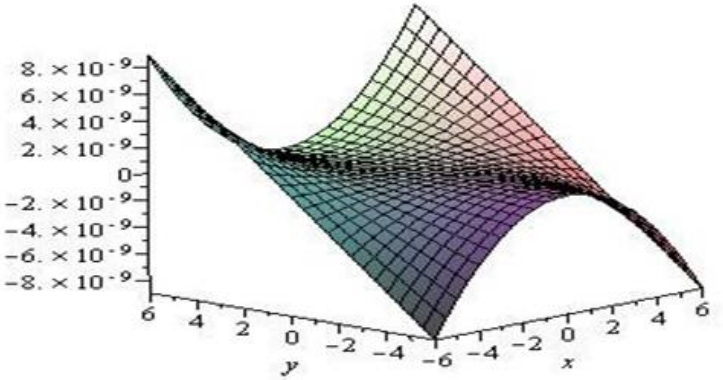


Figure 3. Comparison between Exact Solution $\left(\underline{\delta}(z_1, z_2, r)\right)$ and APx-S Used by HAM $\left(\underline{\delta}_{APx-S[10]}(s, t, r, -1.0)\right)$ by Metric D for Example 1

$$\frac{2}{3}x_1{}^2x_2\left(1-\cos\left(\frac{1}{2}\right)\right)(4-r^3-r)\left[\frac{8}{7}+h\left[1+\sum_{m=2}^{n}\frac{(8+7h)^{m-1}}{2^{3m-3}}\right]\right]$$

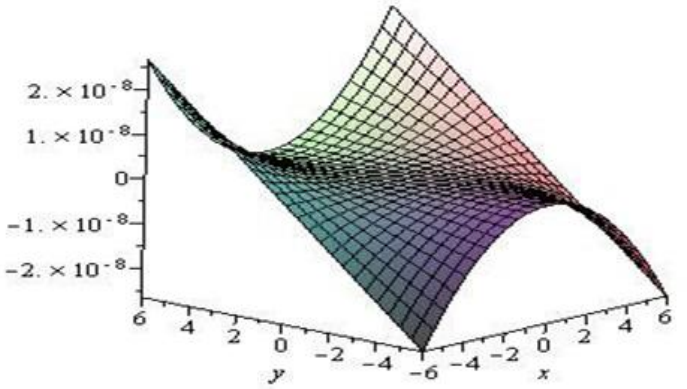


Figure 4. Comparison between Exact Solution $\left(\bar{\delta}(z_1, z_2, r)\right)$ and APx-S Used by HAM $\left(\bar{\delta}_{APx-S[10]}(s, t, r, -1.0)\right)$ by Metric D for Example 1

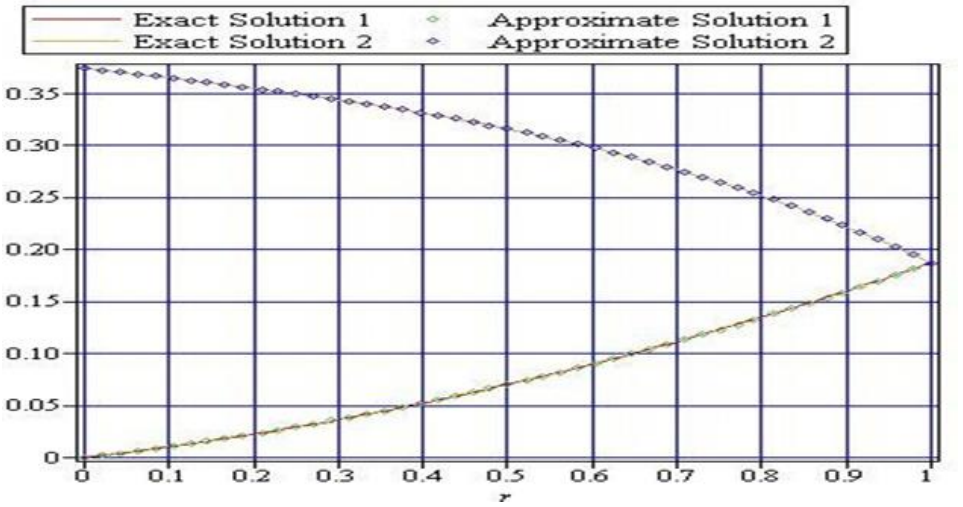


Figure 5. Graphical Representation of Comparison between Exact Solution $\left(\underline{\delta}(z_1, z_2, r), \bar{\delta}(z_1, z_2, r)\right)$ and APx-S Used by HAM $\left(\underline{\delta}_{APx-S[10]}(s, t, r, -1.0), \bar{\delta}_{APx-S[10]}(s, t, r, -1.0)\right)$ for Example1

Table 1. Numerical Results of Example 1with HAM.

r	Exact Solution $\left(\underline{\delta}(z_1, z_2, r), \bar{\delta}(z_1, z_2, r)\right)$	HAM for $z_1 = 0.3, z_2 = 0.6, h = -1.1$	Absolute Error for $z_1 = 0.3, z_2 = 0.6, h = -1.1$
0.0	(0.00000000000,0.3747706607)	(0.00000000000,0.3747706607)	$(0.0\times10^{-00}, 0.0\times10^{-00})$
0.1	(0.01030619317,0.3653077015)	(0.01030619317,0.3653077015)	$(1.5\times10^{-12}, 3.0\times10^{-12})$
0.2	(0.02248623964,0.3552825864)	(0.02248623964,0.3552825863)	$(3.4\times10^{-12}, 6.3\times10^{-12})$
0.3	(0.03654013942,0.3441331592)	(0.03654013942,0.3441331592)	$(6.0\times10^{-12}, 1.0\times10^{-11})$
0.4	(0.05246789250,0.3312972640)	(0.05246789250,0.3312972640)	$(9.1\times10^{-12}, 1.5\times10^{-11})$
0.5	(0.07026949887,0.3162127450)	(0.07026949889,0.3162127450)	$(1.3\times10^{-11}, 2.0\times10^{-11})$
0.6	(0.08994495856,0.2983174459)	(0.08994495858,0.2983174459)	$(1.7\times10^{-11}, 2.7\times10^{-11})$
0.7	(0.11149427160,0.2770492110)	(0.11149427150,0.2770492109)	$(2.2\times10^{-11}, 3.5\times10^{-11})$
0.8	(0.13491743780,0.2518458840)	(0.13491743790,0.2518458839)	$(2.7\times10^{-11}, 4.4\times10^{-11})$
0.9	(0.16021445740,0.2221453092)	(0.16021445750,0.2221453091)	$(3.3\times10^{-11}, 5.6\times10^{-11})$

A formal residual error analysis was performed to further support the accuracy and reliability of the proposed HAM technique. The residual error uniformly approaches zero over the entire domain, as the approximation order n increases from 3 to 10, confirming the rapid convergence of the solution series. Furthermore, comparison with other methods such as the HPM shows that although this approach has been successful in capturing

the system behaviour, HAM offers a clear advantage thanks to its embedded convergence control parameter h. In terms of computational efficiency and error reduction, this parameter/approach enables the user to optimize the convergence rate and achieve remarkable accuracy.

Table 2. Numerical Results of Example 1with HPM.

r	$\left(\underline{\delta}(z_1,z_2,r),\bar{\delta}(z_1,z_2,r)\right)$ Exact solution	$z_1=0.3,z_2=0.6,$ For HPM	$z_1=0.3,z_2=0.6,$ For, absolute error
0.0	(0.00000000000,0.3747706607)	(0.00000000000,0.3749023019)	$(0.0\times10^{-00},1.0\times10^{-10})$
0.1	(0.01030619317,0.3653077015)	(0.01030981330,0.3654360188)	$(2.4\times10^{-12},9.8\times10^{-11})$
0.2	(0.02248623964,0.3552825864)	(0.02249413811,0.3554073822)	$(5.3\times10^{-12},9.6\times10^{-11})$
0.3	(0.03654013942,0.3441331592)	(0.03655297444,0.3442540388)	$(8.6\times10^{-12},9.3\times10^{-11})$
0.4	(0.05246789250,0.3312972640)	(0.05248632227,0.3314136349)	$(1.2\times10^{-11},9.1\times10^{-11})$
0.5	(0.07026949887,0.3162127450)	(0.07029418162,0.3163238173)	$(1.6\times10^{-11},8.8\times10^{-11})$
0.6	(0.08994495856,0.2983174459)	(0.08997655247,0.2984222323)	$(2.0\times10^{-11},8.4\times10^{-11})$
0.7	(0.11149427160,0.2770492110)	(0.1115334349,0.2771465267)	$(2.6\times10^{-11},7.9\times10^{-11})$
0.8	(0.13491743780,0.2518458840)	(0.1349648287,0.2519343468)	$(3.1\times10^{-11},7.4\times10^{-11})$
0.9	(0.16021445740,0.2221453092)	(0.1602707341,0.2222233395)	$(3.7\times10^{-11},6.7\times10^{-11})$

9. CONCLUSION

The implementation of the Homotopy Analysis Method (HAM) to obtain the approximate solution of Kind-II Double Dimension Fuzzy Fredholm Integral Equations (KII-DD-FF-IEqs) was carried out in this manuscript. The numerical results confirm that HAM offers a reliable method with a high accuracy via the auxiliary parameter h, achieving absolute error on the order of 10^{-11}, clear convergence control through numerical example, and comparative evaluation with existing techniques, such as the homotopy perturbation method. Although our results demonstrate that HAM is a valid, competitive, efficient, and affordable alternative to existing methods for solving the class of problems studied here, further research is needed to determine its broader computational performance on larger families of nonlinear FIEqs. So, this method can be extended for the other types of systems of IEqs, such as nonlinear system of Kind-II or Kind-III double dimension Fuzzy Fredholm, Fuzzy Volterra, and Fuzzy Fredholm-Volterra IEqs. In addition, the method was implemented in the form of algorithm to solve the problems for numerical computations and draw a comparison between the exact and approximate solutions.

Author Contribution

Sajid Hussain: conceptualization, methodology, supervision, writing-original draft. **Muhammad Sajid Rashid**: formal analysis, software. **Javid Shabbir:** methodology, validation, visualization,

resources. **Shafqat Ali**: formal analysis, project administration, validation, visualization. **Samer Parveen**: funding acquisition, data curation, resources, visualization, writing-review & editing.

Muhammad Ameeq: methodology, data curation, validation, visualization, resources.

Conflict of Interest

The authors declare that they have no financial or non-financial conflict of interest in the subject matter or materials discussed in this manuscript.

Data Availability Statement

Data availability is not applicable as no new data was created.

Funding Details

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The authors did not use any type of generative artificial intelligence software for this research except for the graphical abstract.

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