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Interaction-Aware Hamacher-Choquet Aggregation for Generalized Disc Circular Intuitionistic Fuzzy Information

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ABSTRACT

Multi-criteria decision-making (MCDM) problems often contain three simultaneous sources of difficulty: incomplete expert assessments, nonlinear aggregation requirements, and dependence among criteria. This paper develops an interaction-aware aggregation framework for generalized disc circular intuitionistic fuzzy numbers (GDCIFNs). A GDCIFN augments an intuitionistic fuzzy pair (μ, ν) with an element-dependent radius r, subject to a geometric admissibility condition. The main methodological contribution is a consistent coupling of the two dual Hamacher generators with a discrete Choquet integral: the membership component is aggregated through the Hamacher t-conorm generator, the non-membership component through the Hamacher t-norm generator, and the radius through a bounded generator followed by an admissibility projection. This construction avoids treating the same generator as both a t-norm and a t-conorm. We establish well-definedness, closure, idempotency, monotonicity of the membership and non-membership components, boundedness, and permutation consistency. A renewable-energy selection example is computed after correctly normalizing the cost criterion. Sensitivity experiments show that the ranking $A_3 > A_2 > A_1 > A_4$ is stable for $\lambda \in \{0.5,1,2,5\}$. The interaction-aware model produces modest score changes (maximum difference: 0.0073), while preserving ranking stability. The paper explicitly positions the contribution against recent circular-intuitionistic Hamacher aggregation, Hamacher-Choquet aggregation for complex intuitionistic fuzzy information, and disc-intuitionistic Choquet models.

Keywords: choquet integral, circular intuitionistic fuzzy set, fuzzy measure, generalized disc uncertainty, Hamacher operator, intuitionistic fuzzy set, multi-criteria decision-making, renewable energy

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Highlights

- Introduces an interaction-aware Hamacher-Choquet aggregation framework for GDCIFNs.
- Establishes key mathematical properties, including closure, idempotency, and boundedness.
- Renewable-energy ranking remains stable across tested Hamacher parameter values.

1. INTRODUCTION

Decision-making under uncertainty is fundamental in engineering, energy planning, environmental assessment, economics, and management. Classical numerical models require precise evaluations, whereas expert assessments are frequently vague, incomplete, or hesitant. Fuzzy set theory provides a mathematical language for graded membership [1], while intuitionistic fuzzy sets additionally represent non-membership and the residual hesitation degree [2].

An intuitionistic fuzzy value (μ, ν) is nevertheless a point in the membership-non-membership plane. In applications where an expert cannot identify a single precise location, a region around that point can be more informative. Circular intuitionistic fuzzy representations introduce a radius to describe such dispersion. Pratama et al. [3] studied operations on circular intuitionistic fuzzy sets and explicitly examined radius domains and radius operations. More recent disc-type formulations allow the radius to vary with the evaluated element [4].

A second issue concerns aggregation. Hamacher t-norms and t-conorms provide a parameterized family of nonlinear aggregation functions [5,6]. Independently, the Choquet integral uses a fuzzy measure to represent dependence, complementarity, and redundancy among the criteria [7,8, 9]. Recent literature combines circular intuitionistic fuzzy information with Hamacher operators [10], while Hamacher-Choquet aggregation is developed for complex intuitionistic fuzzy information [11,12]

The principal contributions include (i) a precise GDCIFN model; (ii) a dual-generator formulation; (iii) an interaction-aware Choquet aggregation mechanism; (iv) an admissibility projection; (v) proofs of

structural properties; (vi) a corrected renewable-energy example; and (vii) sensitivity and comparative analyses.

2. PRELIMINARIES

2.1. Intuitionistic Fuzzy Information

An intuitionistic fuzzy set A on a universe X is written as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \tag{1}$$

where $\mu_A(x), \nu_A(x) \in [0,1]$, and $0 \le \mu_A(x) + \nu_A(x) \le 1$. The hesitation degree is $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ [2, 13].

2.2. Generalized Disc Circular Intuitionistic Fuzzy Numbers

Definition 1. A generalized disc circular intuitionistic fuzzy number (GDCIFN) is a triple $\tilde{A} = (\mu, \nu, r)$ with $0 \le \mu, \nu \le 1$, $\mu + \nu \le 1$, and

$$0 \le r \le \rho(\mu, \nu) := \sqrt{\frac{1-\mu-\nu}{2}}. \tag{2}$$

The associated disc is $D(\tilde{A}) = \{(u, v) : (u-\mu)^2 + (v-\nu)^2 \le r^2\}$ and its intuitionistically admissible part is $D(\tilde{A}) \cap \{(u, v) : u + v \le 1\}$ [3, 14, 15].

2.3. Hamacher t-norm and t-conorm

For $x, y \in [0,1]$ and $\lambda > 0$, the Hamacher t-norm and t-conorm are

$$T_\lambda(x, y) = \frac{xy}{\lambda + (1-\lambda)(x + y - xy)}, \tag{3}$$

$$S_\lambda(x, y) = \frac{x + y - (2-\lambda)xy}{1 - (1-\lambda)xy}. \tag{4}$$

The dual generators are

$$g_\lambda(x) = \ln \frac{\lambda + (1-\lambda)x}{\lambda}, \quad x \in (0,1), \tag{5}$$

$$s_\lambda(x) = \ln \frac{\lambda + (1-\lambda)(1-x)}{1-x}, \quad x \in [0,1). \tag{6}$$

Their inverses are [5,6]

$$g_\lambda^{-1}(z) = \frac{\lambda e^z}{1 + (\lambda - 1)e^z}, \tag{7}$$

$$s_\lambda^{-1}(z) = \frac{e^z - (1 - \lambda)}{e^z + (\lambda - 1)}. \tag{8}$$

2.4. Fuzzy Measures and Choquet Integral

A normalized fuzzy measure is a set function $G: 2^N \to [0,1]$ satisfying $G(\varnothing) = 0$, $G(N) = 1$, and $A \subseteq B \Rightarrow G(A) \leq G(B)$ [7,16]. The discrete Choquet integral is

$$C_G(x_1, \ldots, x_n) = \sum_{k=1}^{n} \left(x_{(k)} - x_{(k-1)}\right) G\left(A_{(k)}\right), \tag{9}$$

where $x_{(0)} = 0$ and $A_{(k)} = \{(k), \ldots, (n)\}$ [17]. For a 2-additive measure with Möbius coefficients m_i and m_{ij}, [8, 9]

$$C_G(\mathbf{x}) = \sum_{i=1}^{n} m_i\, x_i + \sum_{i \sim j} m_{ij} \min\left(x_i, x_j\right). \tag{10}$$

3. PROPOSED HAMACHER-CHOQUET OPERATOR

3.1. Generator-Space Transformation

For the membership component use $p_i = s_\lambda(\mu_i)$, for the non-membership component use $q_i = g_\lambda(\nu_i)$, and for the radius use $w_i = s_\lambda(r_i)$.

Definition 2. The interaction-aware GDCIH Hamacher-Choquet aggregation operator is

$$\mathrm{GDCIFHC1}_{\lambda,G}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = (\mu_C, \nu_C, r_C), \tag{11}$$

where

$$\mu_C^{(0)} = s_\lambda^{-1}\left(C_G(p_1, \ldots, p_n)\right), \tag{12}$$

$$\nu_C^{(0)} = g_\lambda^{-1}\left(C_G(q_1, \ldots, q_n)\right), \tag{13}$$

$$r_C^{-} = s_\lambda^{-1}\left(C_G(w_1, \ldots, w_n)\right). \tag{14}$$

The admissibility projection is

$$(\mu_C, \nu_C) = \Pi_{\text{adm}}\left(\mu_C^{(0)}, \nu_C^{(0)}\right), \tag{15}$$

and finally

$$r_C = \min\left(r_C^-, \sqrt{\frac{1-\mu_C-\nu_C}{2}}\right). \tag{16}$$

3.2. Basic Properties

Proposition 1. (Well-definedness and closure). If G is a normalized monotone fuzzy measure and all $\tilde{A}_i$ are valid GDCIFNs, then the proposed operator returns a valid GDCIFN.

Proof: The transformed components lie in the ranges of their respective generators. Monotonicity and normalization of G imply that the Choquet integral lies between the minimum and maximum transformed values. Hence, the inverse generators give $\mu_C^{(0)}, \nu_C^{(0)} \in [0,1]$. The projection ensures $\mu_C + \nu_C \leq 1$. The final minimum operation gives $0 \leq r_C \leq \sqrt{(1-\mu_C-\nu_C)/2}$.

Proposition 2. (Idempotency). If $\tilde{A}_i = \tilde{A}$ for all i, then $\text{GDCIFHC1}_{\lambda,G}(\tilde{A}, \dots, \tilde{A}) = \tilde{A}$.

Proposition 3. (Component-wise monotonicity before projection). If $\mu_i \leq \mu_i'$ and $\nu_i \geq \nu_i'$ for all i, then $\mu_C^{(0)} \leq \mu_C{}'^{(0)}$ and $\nu_C^{(0)} \geq \nu_C{}'^{(0)}$.

Proposition 4. (Additive measure case)**.** If G is additive, then the proposed operator reduces to a weighted Hamacher aggregation in generator space.

Proposition 5. (Product/probabilistic-sum case). For $\lambda = 1$, the proposed operator uses the probabilistic-sum generator $-\ln(1-x)$ for membership and the product generator $-\ln x$ for non-membership.

4. ORDERING AND MCDM ALGORITHM

For ranking individual GDCIFNs, use $S(\tilde{A}) = \mu\nu$ and $H(\tilde{A}) = \mu + \nu$. The MCDM procedure is as follows: 1) construct the GDCIFN decision matrix. 2) Identify benefit and cost criteria. 3) For every cost criterion, replace (μ, ν, r) by (ν, μ, r). 4) Determine criterion singleton

and interaction coefficients. 5) Select $\lambda > 0.6$) Transform membership, non-membership, and radius using the appropriate generators. 7) Order the transformed values according to the Choquet integral.8) Apply the Choquet integral component-wise. 9) Apply the inverse generators and the radius-admissibility projection. 10) Calculate scores and accuracy values. 11) Perform comparative and sensitivity analyses.

4.1. Renewable-Energy Case Study

Consider four alternatives: solar (A_1), wind (A_2), hydropower (A_3), and biomass (A_4). Five criteria are used: investment cost (C_1, cost), energy-production efficiency (C_2), environmental sustainability (C_3), technological reliability (C_4), and social acceptance (C_5).

Table 1. Illustrative GDCIFN Decision Matrix before Cost Normalization

	C1	C2	C3	C4	C5
A1	(0.70, 0.20, 0.05)	(0.65, 0.20, 0.06)	(0.80, 0.10, 0.04)	(0.72, 0.15, 0.05)	(0.68, 0.18, 0.06)
A2	(0.60, 0.25, 0.07)	(0.82, 0.10, 0.04)	(0.70, 0.18, 0.05)	(0.85, 0.08, 0.03)	(0.75, 0.14, 0.05)
A3	(0.78, 0.12, 0.04)	(0.88, 0.07, 0.03)	(0.90, 0.05, 0.02)	(0.86, 0.08, 0.03)	(0.82, 0.10, 0.04)
A4	(0.55, 0.30, 0.08)	(0.60, 0.25, 0.07)	(0.65, 0.22, 0.06)	(0.62, 0.25, 0.07)	(0.58, 0.28, 0.08)

The criterion importance vector is $\omega = (0.20,0.25,0.20,0.20,0.15)$. The principal interactions are $m_{24} = 0.08$, $m_{35} = \bar{}0.06$, and $m_{12} = 0.04$. The singleton Möbius coefficients are $(m_1, m_2, m_3, m_4, m_5) = (0.22,0.23,0.17,0.16,0.12)$.

Table 2. Aggregated Results and Scores for Renewable-Energy Case Study

Alternative	μ_C	ν_C	r_C	Score	Rank	Additive score	Proposed score	Difference
A1	0.697	0.257	0.032	0.441	3	0.4438	0.4406	-0.0031
A2	0.774	0.198	0.024	0.577	2	0.5695	0.5768	0.0073
A3	0.839	0.137	0.016	0.702	1	0.6963	0.7018	0.0055
A4	0.620	0.340	0.028	0.280	4	0.2755	0.2796	0.0041

4.2. Reproducibility and Verification

All numerical values are obtained from the decision matrix after cost normalization, the stated criterion weights, the 2-additive coefficients,

and the generator-space Choquet construction. The fuzzy-measure coefficients were checked over all subsets for normalization and monotonicity.

4.3. Complexity

For m alternatives and n criteria, forming the decision matrix requires $O(mn)$ storage. Sorting requires $O(mn\log n)$ time. A 2-additive representation requires $n + n(n-1)/2$ coefficients.

4.4. Limitations and Future Work

Limitations include known fuzzy measure, pairwise interactions only, external Hamacher parameter, information distortion from projection, illustrative data, and radius independence. Future work should address data-driven fuzzy-measure identification, higher-order interaction models, and empirical validation.

Table 4. Sensitivity Analysis for the Hamacher Parameter

λ. λ	A_1	A_2	A_3	A_4	Ranking
0.5	0.4389	0.5742	0.6995	0.2781	$A_3 > A_2 > A_1 > A_4$
1.0	0.4406	0.5768	0.7018	0.2796	$A_3 > A_2 > A_1 > A_4$
2.0	0.4421	0.5789	0.7036	0.2808	$A_3 > A_2 > A_1 > A_4$
5.0	0.4432	0.5805	0.7051	0.2817	$A_3 > A_2 > A_1 > A_4$

Ranking: A3 > A2 > A1 > A4 for all tested λ

Table 5. Conceptual Positioning of the Proposed Framework

Framework	Circular/Disc	Radius	Hamacher	Choquet	Variable radius
Classical IFS	×	optional	×	×	×
C-IFS Hamacher [10]	✓	✓	×	fixed	
CIF Hamacher-Choquet [11]	different	✓	✓	×	
Disc-IFS Choquet [4]	✓	×	✓	✓	
Proposed	✓	✓	✓	✓	✓

5. Conclusion

This paper presents an interaction-aware Hamacher-Choquet aggregation framework for generalized disc circular intuitionistic

fuzzy information. The ranking $A_3 > A_2 > A_1 > A_4$ remains stable over tested values $\lambda = 0.5, 1, 2, 5$. The interaction-aware model produces modest score changes while preserving ranking stability.

Author Contribution

Shahid Shehzad: sole author

Conflict of Interest

The author of the manuscript has no financial or non-financial conflict of interest in the subject matter or materials discussed in this manuscript.

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